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Construction and analysis of anomaly-free supersymmetric $SO(2N)/U(N)$ σ-models

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Abstract This paper discusses a procedure for the consistent coupling of gauge- and matter superfields to supersymmetric sigma-models on symmetric coset spaces of Kähler type. We exhibit the finite isometry transformations and the corresponding Kähler transformations. These lead to the construction of a generalized type of Killing potentials. In certain cases a charge quantization condition needs to be imposed to guarantee the global existence of a line bundle on a coset space. The results are applied to the explicit construction of sigma-models on cosets $SO(2N)/U(N)$. Only a finite number of these models can consistently incorporate matter in representations descending from the spinorial representations of $SO(2N)$. We investigate in detail some aspects of the vacuum structure of the gauged $SO(10)/U(5)$ theory, with surprising results: the fully gauged minimal anomaly-free model is shown be singular, as the kinetic terms of the quasi-Goldstone fermions vanish in the vacuum. Gauging only the linear isometry group $SU(5) \times U(1)$, or one of its subgroups, can give a physically well-behaved theory. With gauged $U(1)$ this requires the Fayet-Iliopoulos term to take values in a specific limited range.

1 Introduction

$N = 1$ supersymmetry in 4-D space-time is likely to be a major ingredient of effective field theories of fundamental interactions at energies below or near the Planck scale. When restricted to theories at most quadratic in field gradients, the full spectrum of models is characterized by the field content (e.g., the spectrum of scalar and vector multiplets), which includes fixing the group of local gauge symmetries and their representations; and furthermore by the choice of three functions of the scalar multiplets, the Kähler potential, the superpotential and the holomorphic kinetic functions of the gauge fields. Even with such a restricted set of choices, a surprisingly rich variety of structures can be realized; as yet their classification is far from complete.

In this paper we discuss supersymmetric σ-models on Kähler coset spaces G/H. Such models have been studied previously by various groups of authors [1]; for some reviews see ref.[2, 3, 4]. They have been considered in the context of non-standard superunification models, of effective low-energy models for gauge theories in the strong-coupling limit, or as models for string-inspired low-energy phenomenology. Supersymmetric σ-models of various kinds are also part of supergravity theories. In $N = 1$ supergravity Kähler type models are of interest because they can realize many varieties of non-linear symmetries on chiral fermions. In extended supergravity σ-models are a basic part of the theory, cf. the non-compact models on $SU(1,1)/U(1)$ in $N = 4$, and on $E_{7\,(+7)}/SU(8)$ in $N = 8$ supergravity in 4-D space-time.

Supersymmetry requires the target space of $N = 1$ scalar superfield theories in $D = 4$ space-time to be a complex manifold of the Kähler type [5]. For a coset model to be Kähler imposes special conditions on the groups G and H; in particular, the stability group H always factorizes so as to possess at least one commuting $U(1)$ subgroup. The more special *symmetric cosets* with such a structure include the Grassmannian models on $U(N,M)/U(N) \times U(M)$, the orthogonal unitary coset models on $SO(2N)/U(N)$, as well as models on exceptional cosets like $E_6/SO(10) \times U(1)$. A non-symmetric model of phenomenological interest is for example the supersymmetric version of $E_8/SO(10) \times SU(3) \times U(1)$.

By themselves, homogeneous supersymmetric coset-models are known to be inconsistent quantum field theories, because of the appearance of anomalies in the holonomy group [6]. These can not be compensated by Wess-Zumino type modifications [4]. A particular solution to this problem has been proposed in [7, 8], involving a procedure known as Goldstone boson doubling. This procedure takes a complexification of the broken isometry group as the starting point for the construction of field-theory models. Alternatively, in [10] it was proposed to cancel such anomalies by coupling additional (chiral) matter superfields in non-trivial representations of the isometry group of the σ-model. More details of this procedure have since been worked out in [11, 12, 14]. We wish to stress, that although the two approaches have rather different starting points, they are not

mutually exclusive.

Continuing our line of investigation, in this paper we describe several new results. First, we perform a quite general analysis of the global aspects of the geometry and isometries of Kähler-type coset manifolds. From the results we derive a better and more detailed understanding of the consistency conditions on the bundles which can be constructed over such manifolds. As these bundles can be interpreted as target spaces of fields coupled to the σ-model, the consistency conditions have direct implications for the existence of interacting field theories constructed on the basis of pure coset models. We apply the results of this analysis to the particular case of symmetric orthogonal unitary cosets on $SO(2N)/U(N)$. We show that only a finite number of these models can be consistent when coupled to matter superfields with $U(N)$ quantum numbers reflecting spinorial representations of $SO(2N)$. Among these are in particular the ones based on $SO(10)/SU(5) \times U(1)$, with matter in representations descending from the $\underline{16}$ of $SO(10)$, which are interesting for phenomenological applications.

This paper is structured as follows. In section 2 we review some basic aspects of Kähler geometry and its role in supersymmetric scalar field theories in $D = 4$ space-time. We discuss the symmetries of these models in the geometric language of Killing vectors (generating isometries), which represent infinitesimal, but generally non-linear, transformations. A procedure for coupling chiral superfields in other representations of the isometry group, first described in [9, 10, 11], is reviewed emphasizing its role in anomaly-cancelation. In section 3 we present the construction of non-linear realizations of the $SL(N+M;\mathbb{C})$ starting from the approach of ref. [15]. By imposing certain constraints on the group elements one obtains non-linear representations of various classical groups, like $SU(N+M)$, $SO(2N)$, or $USp(2N)$ and their non-compact relatives, in finite form. We discuss the realization of the non-linear transformations on various types of bundles over the manifold, and examine the consistency conditions to be satisfied. In the context of field theory, this results in the quantization of $U(1)$ charges for matter fields coupled to the σ-model. In section 4 we turn specifically to non-linear realizations of $SO(2N)$. The bundles of interest for supersymmetric field theory applications are constructed. We use the conditions for existence of these bundles to examine the possibility of cancelling anomalies in section 4.5. We identify $U(N)$-bundles over the $SO(2N)/U(N)$ cosets which together build a spinor representation of $SO(2N)$. It is shown that only a finite number of spinor models can be made anomaly-free. We finish in section 5 by discussing a number of physical aspects of the model on $SO(10)/U(5)$, like internal and supersymmetry breaking. A rather surprising result, which generalizes to other coset models, is that the model with fully gauged $SO(10)$ is singular: the kinetic terms of the Goldstone superfields vanish in the vacuum. Gauging the linear subgroup $U(5)$ can give consistent models, but only in a range of non-zero values of the Fayet-Iliopoulos term. Finally, the appendices describe some mathematical details of our constructions.

2 σ-models on Kähler manifolds

The kinetic terms of complex scalar fields $(z^\alpha, z^{\underline{\alpha}})$ on Kähler manifolds are of the form

$$\mathcal{L}_{kin} = - \sqrt{-g}\, g^{\mu\nu}\, G_{\alpha\underline{\alpha}}(z, \bar{z})\, \partial_\mu z^\alpha\, \partial_\nu \bar{z}^{\underline{\alpha}}, \tag{1}$$

with the spacetime metric $g_{\mu\nu}$ and the target-space metric subject to the Kähler condition

$$G_{\alpha\,\underline{\alpha},\beta} = G_{\beta\,\underline{\alpha},\alpha}. \tag{2}$$

Locally on the target manifold the condition is satisfied by deriving the metric from a Kähler potential:

$$G_{\alpha\,\underline{\alpha}} = K_{,\alpha\,\underline{\alpha}}. \tag{3}$$

In terms of the Kähler two-form

$$\omega(K) = -i\, K_{,\alpha\underline{\alpha}}\, d\bar{z}^{\underline{\alpha}} \wedge dz^\alpha \tag{4}$$

equation (2) can be written as

$$d\omega(K) = 0. \tag{5}$$

Obviously, the Kähler potential is defined by this equation only up to holomorphic terms:

$$\tilde{K}(z, \bar{z}) = K(z, \bar{z}) + F(z) + \bar{F}(\bar{z}). \tag{6}$$

As a result, if two complex local co-ordinate charts $\{z_i\}$ and $\{z_j\}$ have non-empty overlap, the Kähler potentials in the charts are generally related by

$$K_i(z_i, \bar{z}_i) = K_j(z_j, \bar{z}_j) + F_{(ij)}(z_j) + \bar{F}_{(ij)}(\bar{z}_j). \tag{7}$$

In this paper we focus on the class of Kähler manifolds formed by coset spaces G/H. Such cosets are Kähler manifolds if H is the centralizer of a torus in G; that is, the stability group H contains one or more $U(1)$ subgroups commuting with the rest of H. A general procedure for constructing a Kähler potential for these coset manifolds was developed by the authors of ref. [15]. It is discussed in some detail in the context of our applications below.

Scalar lagrangeans of the type (1) can be extended to incorporate $N = 1$ Poincaré supersymmetry, by taking the complex fields z^α to be the scalar components of chiral superfields Φ^α; we denote its chiral fermion components by ψ_L^α. In Minkowski space, the component lagrangean is [5]

$$\begin{aligned} \mathcal{L}_{chiral} &= \int d^4\theta\, K(\Phi, \bar{\Phi}) && (8) \\ &= -\, G_{\alpha\underline{\alpha}}(z, \bar{z}) \left[\partial^\mu z^\alpha\, \partial_\mu \bar{z}^{\underline{\alpha}} + \bar{\psi}_L^{\underline{\alpha}} \overleftrightarrow{\not{D}}\, \psi_L^\alpha \right] + \frac{1}{4} R_{\alpha\underline{\alpha}\beta\underline{\beta}}\, \bar{\psi}_L^{\underline{\alpha}} \gamma^\mu \psi_L^\alpha\, \bar{\psi}_L^{\underline{\beta}} \gamma_\mu \psi_L^\beta, \end{aligned}$$

where the covariant derivative and curvature tensor are those of the Kähler manifold.

In general, Kähler metrics admit a set of holomorphic isometries $R_i^{\alpha}(z)$, with conjugates $\bar{R}_i^{\underline{\alpha}}(\bar{z})$, satisfying the Killing equation

$$R_{i\,\underline{\alpha},\alpha} + \bar{R}_{i\,\alpha,\underline{\alpha}} = 0. \tag{9}$$

These isometries define infinitesimal symmetry transformations on the manifold:

$$\delta z^{\alpha} = z'^{\,\alpha} - z^{\alpha} = \theta^i\, \delta_i z^{\alpha} = \theta^i\, R_i^{\alpha}(z), \tag{10}$$

with θ^i the parameters of the infinitesimal transformations. As a result, the isometries define a Lie algebra with structure constants $f_{ij}{}^k$ via the Lie derivative by:

$$(\mathcal{L}_{R_i}[R_j])^{\alpha} = R_i^{\beta} R_{j,\beta}^{\alpha} - R_j^{\beta} R_{i,\beta}^{\alpha} = f_{ij}{}^k\, R_k^{\alpha}. \tag{11}$$

The invariance of the metric implies, that under these isometries the Kähler potential generally is invariant modulo holomorphic functions, as in eq. (6):

$$\delta_i K = F_i(z) + \bar{F}_i(\bar{z}). \tag{12}$$

From the Lie-algebra property (11) it follows that one can choose the transformations of the functions $F_i(z)$ to have the property

$$\delta_i F_j - \delta_j F_i = f_{ij}{}^k\, F_k. \tag{13}$$

Equation (9) for holomorphic Killing vectors has a local solution in terms of a set of scalar potentials $M_i(z, \bar{z})$, transforming in the adjoint representation of the Lie-algebra (11):

$$-iM_i = K_{,\alpha} R_i^{\alpha} - F_i, \qquad\qquad \delta_i M_j = f_{ij}{}^k M_k. \tag{14}$$

Supersymmetry generalizes the isometries to transformations of the chiral superfields Φ^{α}:

$$\delta_i \Phi^{\alpha} = R_i^{\alpha}(\Phi). \tag{15}$$

For the chiral fermions this implies the infinitesimal transformation rule

$$\delta_i \psi_L^{\alpha} = R_{i\ ,\beta}^{\alpha}(z)\, \psi_L^{\beta}. \tag{16}$$

In sect. 3 we present the finite form of the transformations (15), of which (10) and (16) are special cases, for a large class of symmetric coset spaces G/H.

As the chiral fermions couple to the connection and the curvature in $\mathcal{L}_{chiral}$, the consistency of the quantum theory is generally spoiled by anomalies [2]. Therefore we extend the model with additional chiral superfields —generically

called *matter* superfields— on which the isometry group is realized, with the representations chosen to cancel the anomalies.

A general procedure for matter coupling has been worked out in [9, 10, 11]; the generalization to supergravity was presented in [12]. The mathematical framework used to construct matter representations of the isometry group of the Kähler manifold is the theory of complex bundles over Kähler manifolds. These bundles are defined locally on the Kähler manifold by sets of complex fields with specific transformation character under the isometries.

The basic pattern is that exhibited by the transformation rule (16) for the chiral fermions. This rule shows how a vector (an element of the tangent bundle) transforms under the isometries. Similarly, one can define a representation transforming as a 1-form (an element of the co-tangent bundle):

$$\delta_i v_\alpha \,=\, -R_{i\,,\alpha}^{\beta}(z)\, v_\beta. \tag{17}$$

More general tranformations are obtained by taking tensor products of the tangent or co-tangent bundles. However, for our applications this is not sufficient. The reason is, that the $U(1)$ charges of such representations are completely fixed in terms of the charge of the scalars z^α: a contravariant holomorphic tensor $t^{\alpha_1\ldots\alpha_p}$ of rank p carries a relative charge p, whereas a covariant holomorphic tensor $s_{\alpha_1\ldots\alpha_k}$ of rank k carries a relative $U(1)$ charge $-k$. But in actual models, if one requires anomaly cancellations with a phenomenologically interesting set of matter superfields, one usually needs a different assignment of $U(1)$ charges. Therefore the spectrum of representations must be extended with bundels which differ from tensor bundles by the assignment of $U(1)$ charges. This is achieved for instance by the introduction of complex line bundles [11].

A line bundle is the target space of a single-component complex scalar field over the manifold. We consider line bundles carrying non-trivial representations of the isometry group; these can be defined locally on the Kähler manifold as complex scalar matter fields $S(x)$ coupled to the σ-model, with the infinitesimal transformation law given by

$$\delta_i S \,=\, F_i(z) S. \tag{18}$$

In the context of supersymmetric field theories such a representation of the isometry group was introduced in [10], and subsequently considered in [13]; it is a representation because of the property (13). From the line-bundle S one can obtain other line bundles with different $U(1)$ weights by taking powers:

$$A \,\equiv\, S^\lambda \quad \Rightarrow \quad \delta_i A \,=\, \lambda F_i(z) A. \tag{19}$$

Furthermore, using the line bundle construction, one can modify the transformation rules of fields in tensor representations of the isometry group. For example, defining

$$T^{\alpha_1\ldots\alpha_p} \,\equiv\, S^\lambda\, t^{\alpha_1\ldots\alpha_p}, \tag{20}$$

the new field T obeys the transformation rule

$$\delta_i T^{\alpha_1 \dots \alpha_p} = \sum_{k=1}^{p} R_{i\ ,\beta}^{\alpha_k} T^{\alpha_1 .. \beta .. \alpha_p} + \lambda F_i T^{\alpha_1 \dots \alpha_p}. \tag{21}$$

In this way the $U(1)$ charges can be adjusted, be it subject to the charge quantization conditions mentioned above.

However with the introduction of the line-bundle we still have not exhausted all possibilities for consistent non-linear realizations of symmetries over Kähler manifolds. Some coset spaces allow factorization of the goldstone-boson transformations and the Kähler metric. Then one can define sub-bundles of the tangent-space bundles, and their line-bundle extensions as well. A general description of matter representations that can be associated with coset spaces can be found in refs. [2, 22]. Examples of this type of structures are presented below.

The bundles introduced here are characterized locally on the Kähler manifold by their transformation properties. An important question is, if these definitions can be extended globally over the manifold. This is always possible for tangent and co-tangent bundles. However, for line bundles (18), this requires in particular that the holomorphic transition functions introduced in (7) satisfy the cocycle condition

$$F_{(ij)} + F_{(jk)} + F_{(ki)} = 2\pi i \mathbb{Z}. \tag{22}$$

Manifolds with this property are known as Kähler-Hodge manifolds [23]; their Kähler forms satisfy the condition

$$\int_{C_2} \omega(K) = 2\pi \mathbb{Z}, \tag{23}$$

for any closed two-cycle C_2.

The existence of the generalized line bundles (19) and (21) often requires the powers λ to satisfy certain integrality conditions: there is a minimal line bundle which by eq. (22) is globally defined and single-valued, and all other line bundles carry integral charges w.r.t. the minimal line bundle. Thus it follows that the $U(1)$ charges of fields transforming as line bundles are quantized [14]. These consistency requirements are discussed in detail in section 3 below.

3 Non-Linear Realization of $SL(N+M,\mathbb{C})$

In this section we discuss the method developed by Bando, Kuramoto, Maskawa and Uehara (BKMU) [15] to obtaining non-linear transformations and Kähler potentials, but here we consider more general transformations of the complex coordinates and we discuss matter coupling in detail. The basis of our construction

is to define a transformation rule for a complex $M \times N$-matrix z under the action of an arbitrary element of special linear group $SL(M+N;\mathbb{C})$. It will become clear below, why we restrict ourselves to the special linear group. As the special linear group contains all (classical) Lie groups as subgroups, this construction can be used to obtain Kähler potentials for kählerian coset spaces based on these groups. It explains why the transformation rules for the different coset spaces based on classical groups are much alike. Given a coset the group of isometries is fixed, but we can still use the full $SL(M+N;\mathbb{C})$ transformations to determine the effect of coordinate redefinitions. In particular for a non-compact coset this allows us to interpolate between two seemingly different representations of its Kähler potential.

Let $g \in SL(M+N;\mathbb{C})$ be an arbitrary element of the special linear group and g^{-1} its inverse, we write

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \alpha^{\dashv} & \beta^{\dashv} \\ \gamma^{\dashv} & \delta^{\dashv} \end{pmatrix}, \tag{24}$$

where α, β, γ and δ are $M\times M$-, $M\times N$-, $N\times M$- and $N\times N$-matrices respectively. The submatrices of the inverse g^{-1} are given by

$$\begin{aligned} \alpha^{\dashv} &= (\alpha - \beta\delta^{-1}\gamma)^{-1}, \qquad \delta^{\dashv} = (\delta - \gamma\alpha^{-1}\beta)^{-1}, \\ \beta^{\dashv} &= -(\alpha - \beta\delta^{-1}\gamma)^{-1}\beta\delta^{-1} = -\alpha^{-1}\beta(\delta - \gamma\alpha^{-1}\beta)^{-1}, \\ \gamma^{\dashv} &= -(\delta - \gamma\alpha^{-1}\beta)^{-1}\gamma\alpha^{-1} = -\delta^{-1}\gamma(\alpha - \beta\delta^{-1}\gamma)^{-1}. \end{aligned} \tag{25}$$

To obtain the infinitesimal transformations, one considers infinitesimal deviations from the unit element of $SL(M+N;\mathbb{C})$

$$g = \begin{pmatrix} \mathbb{1} + u & y \\ x & \mathbb{1} - v \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \mathbb{1} - u & -y \\ -x & \mathbb{1} + v \end{pmatrix}, \tag{26}$$

where u, v, x, y are infinitesimal submatrices and the minus in front of the v is useful later. However in the following we are primarily concerned with finite transformations. A non-linear relalization is found by defining the matrix $\xi(z)$ like the BKMU-parameter by

$$\xi(z) = \begin{pmatrix} \mathbb{1} & 0 \\ z & \mathbb{1} \end{pmatrix} \tag{27}$$

and requiring that

$$\xi(z) \longrightarrow \xi({}^{g}z) = g\xi(z)\hat{h}^{-1}(z;g), \quad \text{with} \quad \hat{h} = \begin{pmatrix} (\hat{h}_+)^{-1} & \hat{h}_0 \\ 0 & \hat{h}_- \end{pmatrix}. \tag{28}$$

We have written $(\hat{h}_+)^{-1}$ instead of $\hat{h}_+$ in the matrix $\hat{h}$ for later convenience, at this stage this is merely notation. We find that z transforms as

$${}^{g}z = (\gamma + \delta z)(\alpha + \beta z)^{-1} = \left(\delta^{\dashv} - z\beta^{\dashv}\right)^{-1}\left(\gamma^{\dashv} - z\alpha^{\dashv}\right). \tag{29}$$

under the action of g and the matrix $\hat{h}$ takes the form

$$\hat{h}(z;g) = \begin{pmatrix} (\hat{h}_+)^{-1} & \hat{h}_0 \\ 0 & \hat{h}_- \end{pmatrix} = \begin{pmatrix} \alpha + \beta z & \beta \\ 0 & (\delta^{\dashv} - z\beta^{\dashv})^{-1} \end{pmatrix}. \tag{30}$$

Notice that it follows from the transformation rule of z that general linear transformations have the same effect as special linear transformations. Under the composition of two transformations g' and g we find using (28) that the non-linear transformation (29) respects this composition ${}^{g'}({}^{g}z) = {}^{g'g}z$ and furthermore we find that

$$\hat{h}_-(z;g'g) = \hat{h}_-({}^{g}z;g')\hat{h}_-(z;g) \quad \text{and} \quad \hat{h}_+(z;g'g) = \hat{h}_+(z;g)\hat{h}_+({}^{g}z;g'). \tag{31}$$

In the following we employ two projector operators $\eta_\pm$ defined by

$$\eta_+ = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{32}$$

These definitions allow us to write $(\hat{h}_+)^{-1} \simeq \hat{h}\eta_+ = \eta_+\hat{h}\eta_+$ and $\hat{h}_- \simeq \eta_-\hat{h} = \eta_-\hat{h}\eta_-$, where the symbol $\simeq$ denotes equality of the left-hand side as the unique non-vanishing submatrix of the right-hand side.

Now let $\mathfrak{J} \in SL(M+N;\mathbb{C})$ be a fixed matrix; its properties we develop along the way. We define the $M \times M$-matrix function of $z, \bar{z}$ by

$$\tilde{\chi}_{\mathfrak{J}}^{-1}(z,\bar{z}) \equiv \eta_+\xi^\dagger(\bar{z})\mathfrak{J}\xi(z)\eta_+ = A + Bz + \bar{z}C + \bar{z}Dz \tag{33}$$

and obtain the transformation property

$$\tilde{\chi}_{\mathfrak{J}}^{-1}(z,\bar{z}) \longrightarrow \tilde{\chi}_{\mathfrak{J}}^{-1}({}^{g}z,{}^{g}\bar{z}) = \hat{h}_+^\dagger(\bar{z};g)\tilde{\chi}_{g^\dagger\mathfrak{J}g}^{-1}(z,\bar{z})\hat{h}_+(z;g). \tag{34}$$

Define the subgroup $SL_{\mathfrak{J}}(M+N;\mathbb{C})$ consisting of elements $g \in SL(M+N;\mathbb{C})$ that leave $\mathfrak{J}$ invariant

$$g^\dagger\mathfrak{J}g = \mathfrak{J} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathfrak{J}^{-1} \equiv \begin{pmatrix} A^{\dashv} & B^{\dashv} \\ C^{\dashv} & D^{\dashv} \end{pmatrix}. \tag{35}$$

Hence if $g \in SL_{\mathfrak{J}}(M+N;\mathbb{C})$, the function

$$K_{\mathfrak{J}}(z,\bar{z}) = \ln\det\tilde{\chi}_{\mathfrak{J}}^{-1}(z,\bar{z}) \tag{36}$$

transforms as a Kähler potential

$$K_{\mathfrak{J}}(z,\bar{z}) \longrightarrow K_{\mathfrak{J}}({}^{g}z,{}^{g}\bar{z}) = K_{\mathfrak{J}}(z,\bar{z}) + F(z;g) + \bar{F}(\bar{z};g), \tag{37}$$

with

$$F(z;g) = \ln\det\hat{h}_+(z;g), \qquad \bar{F}(\bar{z};g) = \ln\det\hat{h}_+^\dagger(\bar{z};g^\dagger). \tag{38}$$

If we want to interpret $K_{\mathfrak{J}}$ as a Kähler potential, $K_{\mathfrak{J}}$ has to be a real function $K_{\mathfrak{J}}(z,\bar{z}) = (K_{\mathfrak{J}}(z,\bar{z}))^{\dagger}$. This only happens iff $\mathfrak{J}$ is Hermitean $\mathfrak{J}^{\dagger} = \mathfrak{J}$. The composition rule for F follows directly from eq. (31)

$$F(z; g'g) = F(z; g) + F({}^{g}z; g'). \tag{39}$$

We define a real finite Killing pre-potential $\mathcal{M}(\bar{z}, z; g')$ by

$$2i\,\mathcal{M}(\bar{z}, z; g') = K(z, {}^{g'}\bar{z}) - K({}^{g'}z, \bar{z}) + F(z; g') - \bar{F}(\bar{z}; g'). \tag{40}$$

It is a function of the group element g', of which the infinitesimal (linearized) form reproduces the standard Killing potentials (14). Using the transformation property of the Kähler potential (37) together with the composition property (39) of F, it follows that $\mathcal{M}(\bar{z}, z; g')$ transforms in the adjoint representation

$$\mathcal{M}({}^{g}z, {}^{g}\bar{z}; g') = \mathcal{M}(z, \bar{z}; g^{-1}g'g). \tag{41}$$

Again, inserting a group element close to the identity, we obtain for the Killing potentials the infinitesimal transformation rule (14).

The metric associated with $K_{\mathfrak{J}}$ can be written as

$$G_{\underline{\alpha}\alpha} d\bar{z}^{\underline{\alpha}} dz^{\alpha} = \mathrm{tr}\left[\tilde{\chi}_{\mathfrak{J}} d\bar{z} \chi_{\mathfrak{J}} dz\right], \tag{42}$$

where we define $\chi_{\mathfrak{J}}$ in analogy of $\tilde{\chi}_{\mathfrak{J}}$ in (33)

$$\chi_{\mathfrak{J}}^{-1}(z,\bar{z}) \equiv \eta_{-}\left(\xi^{\dagger}(\bar{z})\mathfrak{J}\xi(z)\right)^{-1}\eta_{-} = D^{\dashv} - C^{\dashv}\bar{z} - zB^{\dashv} + zA^{\dashv}\bar{z}. \tag{43}$$

This can be shown either by a direct calculation of the metric in the standard way as the second mixed derivative or by first proving this for a block-diagonal $\mathfrak{J}$ and showing that the diagonalization procedure has no effect on the metric (42). This is easy as under the action of $g \in SL_{\mathfrak{J}}(M, N)$ the differential dz transforms as

$$dz \longrightarrow^{g}(dz) = \hat{h}_{-}(z; g) dz \hat{h}_{+}(z; g) = (\delta^{\dashv} - z\beta^{\dashv})^{-1} dz (\alpha + \beta z)^{-1}, \tag{44}$$

and $\tilde{\chi}_{\mathfrak{J}}, \chi_{\mathfrak{J}}$ transform as eq. (34) and as

$$\chi_{\mathfrak{J}}^{-1}(z,\bar{z}) \longrightarrow \chi_{\mathfrak{J}}^{-1}({}^{g}z, {}^{g}\bar{z}) = \hat{h}_{-}(z; g)\chi_{\mathfrak{J}}^{-1}(z,\bar{z})\hat{h}_{-}^{\dagger}(\bar{z}; g). \tag{45}$$

Hence it follows that (42) is invariant.

Until this point the matrix $\mathfrak{J}$ used in the definitions (33) and (43) of $\tilde{\chi}_{\mathfrak{J}}$ and $\chi_{\mathfrak{J}}$ can be any Hermitean matrix of $SL(M+N;\mathbb{C})$. However if we want to use the invariants (58) as Kähler potentials for supersymmetric model building, the resulting kinetic terms have to be positive definite. By going to the unitary gauge ($\bar{z} = z^{T} = 0$), we infer that both A and $D^{\dashv}$ have to be sign definite. (Of course an overall sign can be compensated by an appropriate minus sign.) On the other

hand using a unitary transformation, we can diagonalize $\mathfrak{J}$ with real eigenvalues λ_i. If this is followed by an appropriate scale transformation of the coordinates and possibly some relabeling, we bring the matrix $\mathfrak{J}$ into the canonical form

$$\mathfrak{J} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \eta\mathbb{1} \end{pmatrix}, \qquad \eta = \pm 1. \tag{46}$$

This shows that we can restrict $SL_{\mathfrak{J}}(M+N;\mathbb{C})$ to $SU_\eta(M,N)$ when we want to study the isometries of the metrics $\tilde{\chi}_{\mathfrak{J}}$, $\chi_{\mathfrak{J}}$ or the Kähler potential $K_{\mathfrak{J}}$. Here $\eta = +1$ refers to the compact special unitary group $SU(M+N)$, while $\eta = -1$ refers to the non-compact version. We assume from now on that we have chosen this canonical form of $\mathfrak{J}$ and consider $SU_\eta(M,N)$ only. Notice that by putting further restrictions on the group elements g we can reduce the isometry group to a subgroup of $SU(M,N)$, such as $SO(2N)$ or $USp(N)$. The form of the metrics and Kähler potential does not change under this; they always take the form

$$\begin{gathered} \tilde{\chi}_\eta(z,\bar{z}) = \left(\mathbb{1} + \eta \bar{z} z\right)^{-1}, \quad \chi_\eta(z,\bar{z}) = \left(\mathbb{1} + \eta z \bar{z}\right)^{-1}, \\ K_\eta(z,\bar{z}) = \eta \ln\det \tilde{\chi}_\eta^{-1} = \eta \ln\det \chi_\eta^{-1} \end{gathered} \tag{47}$$

in the canonical basis. However this leads to restrictions on the coordinates z as we see later: that is the coordinates z parameterize a submanifold of $SU_\eta(M,N)/S[U(M)\times U(N)]$. Even though the $SL(M+N;\mathbb{C})$ group is not the isometry group, it is still worthwhile to know its action on the fields, as it can be used to describe field redefinitions.

We give an example of this now. In the previous analysis we used that we can set B and C in the matrix $\mathfrak{J}$ to zero by a unitary transformation. Sometimes we can also do the opposite: set A and D to zero. To analyze the situation we start with $\mathfrak{J}$ in the canonical form and perform an arbitrary transformation g of $SL(M+N;\mathbb{C})$ on it

$$g^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \eta\mathbb{1} \end{pmatrix} g = \begin{pmatrix} \bar{\alpha}\alpha + \eta\,\bar{\gamma}\gamma & \bar{\alpha}\beta + \eta\,\bar{\gamma}\delta \\ \bar{\beta}\alpha + \eta\,\bar{\delta}\gamma & \bar{\beta}\beta + \eta\,\bar{\delta}\delta \end{pmatrix}. \tag{48}$$

So to remove the A and D entries of this matrix we need to have that

$$\bar{\alpha}\alpha + \eta\,\bar{\gamma}\gamma = 0 \quad \text{and} \quad \bar{\beta}\beta + \eta\,\bar{\delta}\delta = 0. \tag{49}$$

Notice that there is no solution $g \in SL(M+N;\mathbb{C})$ of these equations when $\eta = 1$. On the other hand in the case $\eta = -1$ with $M = N$ we can use

$$g = \frac{i}{\sqrt{2}} \begin{pmatrix} -\mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \tag{50}$$

to bring $\mathfrak{J}$ into the form

$$\mathfrak{J} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{51}$$

Using this matrix $\mathfrak{J}$ we obtain a Kähler potential

$$K_{no-sc} = \ln\det(z+\bar{z}) \tag{52}$$

of the no-scale type [20]. The low energy effective actions for the moduli sectors of string theory often take this form, see for example [25, 27, 26].

Matter coupling is the next topic we discuss. As we want to interpret $SU_\eta(M,N)$ as the symmetry group of the models we construct, this implies that all matter representations should be well defined representations of $SU_\eta(M,N)$. To obtain a section of the tangent bundle, we define the transformation of the tangent space vector T in analogy of (44) by

$${}^{g}T = \hat{h}_-(z;g)T\hat{h}_+(z;g) = (\delta^\dashv - z\beta^\dashv)^{-1}T(\alpha+\beta z)^{-1}. \tag{53}$$

A section C of the cotangent bundle transforms as

$${}^{g}C = \Big(\hat{h}_+(z;g)\Big)^{-1} C \Big(\hat{h}_-(z;g)\Big)^{-1} = (\alpha+\beta z)C(\delta^\dashv - z\beta^\dashv). \tag{54}$$

When we take $g \in SU_\eta(M,N)$, we obtain the following invariants for the sections of the tangent and cotangent bundles

$$\mathrm{tr}\left[\tilde{\chi}_{\mathfrak{J}}\bar{T}\chi_{\mathfrak{J}}T\right] \quad \text{and} \quad \mathrm{tr}\left[(\chi_{\mathfrak{J}})^{-1}\bar{C}(\tilde{\chi}_{\mathfrak{J}})^{-1}C\right]. \tag{55}$$

Next we construct subbundles of the tangent bundle. To do this we notice that the transformation rule (44) for the differential dz factorizes [14, 22]. Using this we define the sections L and R by the transformation rules

$${}^{g}L = \hat{h}_-(z;g)L = (\delta^\dashv - z\beta^\dashv)^{-1}L \quad \text{and} \quad {}^{g}R = R\hat{h}_+(z;g) = R(\alpha+\beta z)^{-1}. \tag{56}$$

To show that these transformations do indeed define consistent bundles we proceed as follows. All manifolds we consider here are submanifolds of Grassmannian coset-space $SU_\eta(M,N)/S[U(M)\times U(N)]$. As this is a homogeneous space we can reach any point on it by a transformation using a group element $g \in SU_\eta(M,N)$. Therefore we can describe all coordinate transformations as actions of elements of $SU_\eta(M,N)$, and the holomorphic transition functions on overlapping complex co-ordinate charts for the bundle of which L is a section are given by elements $\hat{h}_-(z;g)$. The global consistency conditions for this bundle, mentioned in sect. (2), then take the form

$$\begin{gathered}\hat{h}_-(z;e) = \mathbb{1}, \quad \hat{h}_-({}^{g}z;g^{-1}) = \hat{h}_-(z;g)^{-1} \\ \hat{h}_-({}^{g_2g_1}z;g_3)\hat{h}_-({}^{g_1}z;g_2)\hat{h}_-(z;g_1) = \mathbb{1},\end{gathered} \tag{57}$$

when $g_1g_2g_3 = e$, being e the $SU_\eta(M,N)$ identity. The composition property (31) of two group elements show that these conditions are satisfied. Using the

metric of the tangent bundle (42), which factorizes as well, we obtain the following $SU_\eta(M,N)$-invariants

$$\bar{L}\chi_{\mathfrak{J}}L \quad \text{and} \quad R\tilde{\chi}_{\mathfrak{J}}\bar{R}. \tag{58}$$

We will discuss tensor products of these types of matter representations extensively when we consider matter coupling to $SO(2N)/U(N)$.

Until this point our discussion was general, in the sense that we only demanded that we construct isometries of the metrics $\tilde{\chi}_{\mathfrak{J}}$ and $\chi_{\mathfrak{J}}$ without any reference to a particular coset space. We saw that we only obtain isometries of these metrics if we restrict the transformations to be unitary $g \in SU_\eta(M,N)$. It is now easy to describe non-linear realizations of (classic) groups, that are subgroups of $SU_\eta(M,N)$. For this we only have to describe what the group and algebra of the groups look when embedded in the unitary group $SU_\eta(M,N)$. We have summarized our results in table 1. We describe the ingredients of this table which are partly taken from ref. [21]. A complete classification of Kähler cosets can be found in ref. [16]. A discussion on $SO(2N)/U(N)$, $Sp(2N)/U(N)$ cosets can also be found in refs. [18, 17, 19].

The classic groups are either real or complex groups that satisfy certain Hermitean conjugation and transposition properties

$$g^\dagger \mathfrak{J} g = \mathfrak{J} \quad \text{and} \quad g^T \mathfrak{K}\, g = \mathfrak{K} \tag{59}$$

where $\mathfrak{J}$ and $\mathfrak{K}$ are fixed matrices. We discriminate between the unitary (SU), orthogonal (SO), symplectic (Sp) and unitary symplectic (USp) groups. Furthermore, with $\eta = \pm 1$ we make a distinction between compact ($\eta = 1$) and non-compact ($\eta = -1$) groups. We require the maximal subgroups H of these groups to have a compact $U(1)$-factor. For example, we do not consider the non-compact $SO(N,N)$ here, as the non-compact abelian $SO(1,1)$ subgroup corresponds to Lorentz transformations that are not bounded. A compact $U(1)$-factor is needed to ensure that the resulting coset-space is Kähler; because of its importance we give the $U(1)$ embedding explicitly. For the real groups $SO(2N)$ and $Sp(2N)$ the $U(1)$ is not realized in a diagonal way. By making a similarity transformation

$$g_D = V g V^\dagger, \quad g = V^\dagger g_D V \quad \text{and} \quad V = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & i \\ i & \mathbb{1} \end{pmatrix}, \tag{60}$$

using the unitary matrix V, the $U(1)$ is turned into a diagonal form. Here the subscript D is used to indicate that g_D, for example, is considered in the basis where the $U(1)$ is diagonal. As V is unitary, g_D has the same unitary properties as g. However the transposition properties may change

$$g_D^T \mathfrak{K}_D g_D = \mathfrak{K}_D = \left(V^\dagger\right)^T \mathfrak{K}\, V^\dagger. \tag{61}$$

For this it is crucial that we have embedded the real groups $Sp(2N)$ and $SO(2N)$ in the special unitary group $SU(N,N)$ and $SU(2N)$ respectively; else the multiplication with i has no meaning. In the remainder we work in the basis where the $U(1)$-factor is diagonal. We can now represent any element of any of these groups as a unitary matrix $g_D = e^{a_D}$, that is obtained by exponentiating an anti-Hermitean algebra element a_D. The group definition properties (59) can be written down for the algebra elements a_D as well

$$a_D^\dagger = -\mathfrak{J} a_D \mathfrak{J}^{-1} \quad \text{and} \quad a_D^T = -\mathfrak{K}_D a_D \mathfrak{K}_D^{-1}. \tag{62}$$

Using these properties it is possible to give a unique representation of the algebra elements a_D. For the different groups we give this representation in the row of g_D in table 1. Notice that algebra elements of $Sp(2N)$ and $USp(N,N)$ have the same representation in the basis where the $U(1)$ is diagonal; therefore their corresponding cosets are isomorphic. From this representation of the algebra, it is easy to see what the restrictions are on the coset coordinates z for the different coordinates. For the non-compact coset, the coordinates in addition satisfy $\mathrm{tr}(z\bar{z}) < 1$ for the Kähler potential and metrics in (47) to be well defined. Notice that for the USp, Sp and SO cosets the submetrics $\tilde{\chi}_\eta, \chi_\eta$ are each others transposed $\tilde{\chi}_\eta = \chi_\eta^T$.

We now turn to the construction of the minimal complex line bundles. In this discussion we have to make a distinction between the different coset spaces as becomes clear below. Our discussion here is complementary to ref. [14] where general results have been presented, which we apply here to the particular cosets discussed in this section.

A section S of a complex line bundle can be defined to transform as

$${}^g S = \det \hat{h}_+(z;g) S = \det \hat{h}_-(z;g) S. \tag{63}$$

Here we have used that $\det \hat{h}_+ = \det \hat{h}_-$, which follows from (28) since $g \in SU_\eta(M,N)$. The consistency of this complex line bundle follows directly from (57) and the properties of the determinant. To show that we have obtained the minimal line bundle in the compact situation, we have to show that the integral over the corresponding Kähler form

$$\int_{C_2} \omega(K) = 2\pi\, n, \qquad \text{with} \qquad n = \pm 1, \tag{64}$$

when integrated over a generating two-cycle C_2.

We first turn to a Grassmannian coset $SU(M+N)/S[U(M)\times U(N)]$. Let v be the complex coordinate of the stereographic projection of the complex projective line $\mathbb{C}P^1$. We define a generating two-cycle by the embedding of $\mathbb{C}P^1$ in the coset by taking all the coordinates z^{ij} zero except for one which is equal to v. Now since the Kähler potential restricted to this embedding to $\mathbb{C}P^1$ is given by

Group G	$SU_\eta(M,N)$	$USp_\eta(N,N)$	$Sp(2N)$	$SO(2N)$
$\eta =$	± 1	± 1	-1	1
Compact subgroup H	$S[U(M)\times U(N)]$	$U(N)$	$U(N)$	$U(N)$
$g \in$	$SL(M+N;\mathbb{C})$	$SL(2N;\mathbb{C})$	$SL(2N;\mathbb{R})$	$SL(2N;\mathbb{R})$
$g^\dagger \mathfrak{J} g = \mathfrak{J} =$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta 1 \end{pmatrix}$	–	–
$g^T \mathfrak{K} g = \mathfrak{K} =$	–	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$U(1)$ embedding	$\begin{pmatrix} e^{i\frac{N\theta}{P}} & 0 \\ 0 & e^{-i\frac{M\theta}{P}} \end{pmatrix}$	$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
$g_D^T \mathfrak{K}_D g_D = \mathfrak{K}_D =$	–	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$g_D = e^{a_D},\ a_D =$	$\begin{pmatrix} u & -\eta x^\dagger \\ x & -v \end{pmatrix}$	$\begin{pmatrix} u & -\eta x^\dagger \\ x & -u^T \end{pmatrix}$	$\begin{pmatrix} u & x^\dagger \\ x & -u^T \end{pmatrix}$	$\begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix}$
Restrictions	$u^\dagger = -u,\ v^\dagger = -v$ $\operatorname{tr} u = \operatorname{tr} v$	$u^\dagger = -u,\ x^T = x$	$u^\dagger = -u,\ x^T = x$	$u^\dagger = -u,\ x^T = -x$
$z \in G/H, z^{ij} \in \mathbb{C}$	–	$z^T = z$	$z^T = z$	$z^T = -z$

Table 1: This table gives an overview of the (classical) Lie-groups that can be embedded into $SU_\eta(M,N)$. With the parameter η we distinguish between compact ($\eta = 1$) and non-compact ($\eta = -1$) groups. For these Lie-groups the non-linear $SL(M+N;\mathbb{C})$ transformation rules given in this section can be used directly. $P = \gcd(M,N)$ is defined a the greatest-common-divisor of M and N. When the $U(1)$ is not diagonal, we have to perform a special unitary transformation to make it diagonal; when doing so the transposition properties may change. The Hermitean form of an element of the algebra after possible diagonalization is denoted by a_D. The matrices u, v, x are all taken to be complex, their additional properties are given in second last row in the table. The last row summarizes the symmetry properties of the coset coordinate matrices.

$K(z,\bar z)|_{\mathbb{C}P^1} = \ln(1+\bar v v) = K_{\mathbb{C}P^1}(v,\bar v)$, which is the Kähler potential of $\mathbb{C}P^1$ that satisfies $\int_{\mathbb{C}P^1}\omega(K_{\mathbb{C}P^1}) = 2\pi$, it follows that we have obtained a minimal line bundle. Next we discuss the compact $USp(2N)/U(N)$ and $SO(2N)/U(N)$ coset spaces. The coordinates of these spaces satisfy $z^T = z$ resp. $z^T = -z$, see table 1. Therefore it is not possible to set all coordinates to zero except for one, except when one takes this symmetrization into account: $z = \pm z^T = v$. Hence we find in these cases that $K(z,\bar z)|_{\mathbb{C}P^1} = 2\ln(1+\bar v v) = 2K_{\mathbb{C}P^1}(v,\bar v)$, so that $n = 2$ in eq. (64). This implies that the section S is the square of the minimal line bundle. Since the Kähler potential of a coset is unique up to a normalization factor, it follows that a section of a minimal line bundle over $USp(2N)/U(N)$ or $SO(2N)/U(N)$ is given by

$$ {}^{g}S = \Big(\det \hat h_+(z;g)\Big)^{\frac{1}{2}} S = \Big(\det \hat h_-(z;g)\Big)^{\frac{1}{2}} S. \tag{65} $$

The only possible ambiguity for a global definition resides in the square root, it can be removed by using the BKMU-construction with the representation with highest weight that has all its Dynkin label zero except for the Nth one [14].

We now determine the relative charges of the coordinates z, the matter fields L and R, and the sections of the minimal line bundles, using the $U(1)$ embedding presented in table 1. We first discuss the Grassmannian cosets and after that the cosets $USp(2N)/U(N)$ and $SO(2N)/U(N)$. The $U(1)$-factor in $SU_\eta(M,N)$, that is not in $SU(M)\times SU(N)$, can be given by

$$ u_\theta = \begin{pmatrix} e^{-iN\theta/P}\mathbb{1} & 0 \\ 0 & e^{iM\theta/P}\mathbb{1} \end{pmatrix}, \tag{66} $$

where $P = \gcd(M,N)$ is the greatest-common-divisor of M and N. The smallest period of this $U(1)$ is $\theta = 2\pi$, since the integers N/P and M/P are relatively prime by construction. It follows that the coordinates z have charge $(M+N)/P$ in this normalization. For the matter couplings L and R we find the charges N/P resp. M/P. The section of the minimal line bundle has a charge MN/P. For the cosets $USp(2N)/U(N)$ and $SO(2N)/U(N)$ we always obtain integer charges when we choose a slightly different normalization for u_θ given by

$$ u_\theta = \begin{pmatrix} e^{-i2\theta}\mathbb{1} & 0 \\ 0 & e^{i2\theta}\mathbb{1} \end{pmatrix}. \tag{67} $$

In this case L and R have the same charge 2 and the section of the minimal line bundle has charge N, while the charge of the coordinates is 4.

4 $SO(2N)/U(N)$ coset models

We discuss supersymmetric models build using the Kähler geometry of the coset space $SO(2N)/U(N)$. For this we first discuss the decomposition of the $SO(2N)$

algebra into $U(N)$ representations and the vector representation of $SO(2N)$. We discuss the construction of the Kähler potential using the the general BKMU method and related that to our discussion on special linear transformations of section 3. Next we discuss matter representations that can be coupled the supersymmetric σ-model of the coset in a consistent way, heavily relying on the non-linear transformation discussed in section 3. For applications to a chiral spinor representation of $SO(2N)$ later in this article we confine ourselves to the construction of completely anti-symmetric tensor representations with an arbitrary rescaling charge. We discuss their transformation properties and their invariant Kähler potentials that can be used in supersymmetric model building. Some relevant results and conventions have been collected in the appendices.

4.1 $SO(2N)$ Algebra in a $U(N)$ basis

In this section we discuss how the algebra of $SO(2N)$ can be decomposed into $SU(N)\times U(1)$ representations. We split the $SO(2N)$ generators M_{ab} into $SU(N)$ generators $T^i{}_j$, a $U(1)$-factor generator Y and broken generators $X^{ij}, \bar{X}_{ij}$ which are anti-symmetric tensors of $SU(N)$. We first discuss the embedding of $U(N)$ in $SO(2N)$, then we discuss the vector representation; the spinor representation is discussed in appendix A.

The $2N(2N-1)/2$ anti-Hermitean generators $M_{ab} = -M_{ba}$ of $SO(2N)$ satisfy the commutation relations

$$[M_{ab}, M_{cd}] = \delta_{ac}M_{db} - \delta_{bd}M_{ac} - \delta_{ad}M_{cb} + \delta_{bc}M_{ad}. \tag{68}$$

We denote the N^2 generators of $U(N)$ by $U^i{}_j$ $(i,j = 1,\dots,N)$. The remaining $N(2N-1) - N^2 = N(N-1)$ generators form two anti-symmetric tensor representations of $U(N)$: X^{ij} and $\bar{X}_{ij}$, each of dimension $N(N-1)/2$. The $U(N)$ generators satisfy the algebra

$$[U^i{}_j, U^k{}_l] = \delta^i{}_l U^k{}_j - \delta^k{}_j U^i{}_l. \tag{69}$$

We decompose the $SO(2N)$ algebra w.r.t. $U(N)$ by writing the $SO(2N)$ generators M_{ab} using indices $i,j = 1,\dots,N$ as

$$\begin{array}{lcl} M_{ij} & = & \frac{1}{2}(-X^{ij} - \bar{X}_{ij} - U^i{}_j + U^j{}_i), \\ M_{i\,j+N} & = & \frac{i}{2}(X^{ij} - \bar{X}_{ij} - U^i{}_j - U^j{}_i), \\ M_{i+N\,j+N} & = & \frac{1}{2}(X^{ij} + \bar{X}_{ij} - U^i{}_j + U^j{}_i). \end{array} \tag{70}$$

Inversely we can express $U^i{}_j, X^{ij}$ and $\bar{X}_{ij}$ as $U^i{}_j = A^i{}_j + iS^i{}_j$ with

$$A^i{}_j = -\frac{1}{2}\left(M_{ij} + M_{i+N\,j+N}\right), \qquad S^i{}_j = \frac{1}{2}\left(M_{i\,j+N} + M_{j\,i+N}\right) \tag{71}$$

and $X^{ij} = -iQ^{ij} - P^{ij}$ and $\bar{X}_{ij} = iQ^{ij} - P^{ij}$ with

$$P^i{}_j = \frac{1}{2}\left(M_{i\,j} - M_{i+N\,j+N}\right), \qquad Q^i{}_j = \frac{1}{2}\left(M_{i\,j+N} - M_{j\,i+N}\right). \tag{72}$$

The U(1)-factor generator Y in $U(N)$ is defined as minus twice the trace of the $U(N)$ generators

$$Y = -2\sum_i^N U^i{}_i = -i2S^i{}_i = -2iM_{i\,i+N} \tag{73}$$

and the remaining SU(N) generators $T^i{}_j$ are define as the traceless part of $U^i{}_j$

$$T^i{}_j = U^i{}_j + \frac{1}{2N}Y\delta^i{}_j. \tag{74}$$

Using the $U(N)$ generators $U^i{}_j$ and the broken generators X^{ij} and $\bar{X}_{ij}$ the $SO(2N)$ algebra (68) takes the form

$$\begin{aligned}
&[U^i{}_j, U^k{}_l] = \delta^i{}_l U^k{}_j - \delta^k{}_j U^i{}_l, \qquad [X^{ij}, X^{kl}] = [\bar{X}_{ij}, \bar{X}_{kl}] = 0,\\
&[\bar{X}_{ij}, X^{kl}] = -\delta^k{}_i U^l{}_j - \delta^l{}_j U^k{}_i + \delta^l{}_i U^k{}_j + \delta^k{}_j U^l{}_i,\\
&[U^i{}_j, \bar{X}_{kl}] = \delta^i{}_k \bar{X}_{jl} - \delta^i{}_l \bar{X}_{jk},\\
&[U^i{}_j, X^{kl}] = \delta^l{}_j X^{ik} - \delta^k{}_j X^{il}.
\end{aligned} \tag{75}$$

The closure of the algebra can be checked explicitly by computing the Jacobi identities. The $SO(2N)$ generators in this basis carry the following $U(1)$-charges:

$$U(1)\text{-charges of } (Y, T^i{}_j, X^{ij}, \bar{X}_{ij}) = (0, 0,\ 4,\ -4). \tag{76}$$

Here we have chosen the $U(1)$-charges such that they match the convention of Slansky [28].

4.2 The vector representation of SO(2N)

In the vector representation of $SO(2N)$, the generators M_{ab} take the form: $(M_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}$, therefore an element of the $SO(2N)$-algebra reads

$$\Theta = (-a^{ij}A_{ij} - s^{ij}S_{ij}) + (q^{ij}Q_{ij} - p^{ij}P_{ij}) \equiv \begin{pmatrix} a & -s \\ s & a \end{pmatrix} + \begin{pmatrix} -p & q \\ q & p \end{pmatrix}, \tag{77}$$

where a, p, q are $N \times N$ real anti-symmetric matrices and s is a real symmetric $N \times N$ matrix; these matrices define the parameters of the $SO(2N)$-algebra elements. Here we have used the definitions of the algebra elements A, S, P and

Q given in eqs. (71) and (72). The U(1)-factor generator Y (73) in the vector representation takes the form

$$Y = -2i \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{78}$$

Notice that the $U(1)$ generator Y is off-diagonal. However it is more convenient to use a basis in which Y is diagonal. Using a unitary transformation we can diagonalize Y:

$$Y_D \equiv VYV^\dagger = 2 \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \text{with} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -i\mathbb{1} \\ -i\mathbb{1} & \mathbb{1} \end{pmatrix} . \tag{79}$$

We use the subscript notation D on any $2N \times 2N$-matrix A to indicate that A is evaluated in the basis where Y is diagonal. The effect of this similarity transformation on an element Θ of the $SO(2N)$ Lie algebra (77) is given by

$$\Theta_D = VMV^\dagger = \begin{pmatrix} a - is & q - ip \\ q + ip & a + is \end{pmatrix} = \begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix} , \tag{80}$$

where $u = -u^\dagger = a - is$, $u^T = a + is$, $x = q + ip$ and $x^\dagger = -q + ip$. This is coincides with the $SO(2N)/U(N)$ entry in table 1. Notice that in the basis where Y is diagonal, the defining property $g^{-1} = g^T$ of $SO(2N)$ becomes

$$g_D^{-1} = \mathfrak{K} g_D^T \mathfrak{K} \quad \text{with} \quad \mathfrak{K} \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{81}$$

Writing g_D in terms of the submatrices α, β, γ and δ as introduced in eq. (24) this group property can be stated as

$$\begin{pmatrix} \alpha^{\dashv} & \beta^{\dashv} \\ \gamma^{\dashv} & \delta^{\dashv} \end{pmatrix} = \begin{pmatrix} \delta^T & \beta^T \\ \gamma^T & \alpha^T \end{pmatrix} , \tag{82}$$

using the notation (26) for the inverse of g_D. From now on we will only work in the basis where the $U(1)$-charge Y is diagonal, dropping the subscripts D.

4.3 Kähler and Killing Potentials

We now construct the Kähler potential for the coset spaces $SO(2N)/U(N)$ using the BKMU-method [15]. We apply their method to the $2N$ dimensional vector representation of $SO(2N)$. The BKMU-projection η_+ projects (32) on the part of this vector representation with positive Y-charge, which is an N dimensional vector representation of $SU(N)$. The coset spaces $SO(2N)/U(N)$ and $SO^{\mathbb{C}}(2N)/\hat{U}(N)$, with $SO^{\mathbb{C}}(2N)$, the complexification of $SO(2N)$, are isomorphic because $\hat{U}(N)$ is defined as the group generated by all generators of $U(N)$

together with the broken generators X^{ij} over the complex numbers. The representative $\xi(z) \in SO^{\mathbb{C}}(2N)/\hat{U}(N) \cong SO(2N)/U(N)$ of the equivalence class $\xi(z)\hat{U}(N)$ is given in terms of the $\frac{1}{2}N(N-1)$ coordinates z^{ij} of $SO(2N)/U(N)$ by

$$\xi(z) = \exp Z = \begin{pmatrix} \mathbb{1} & 0 \\ z & \mathbb{1} \end{pmatrix}, \qquad Z = -\frac{i}{2} z^{ij} \bar{X}_{ij}. \tag{83}$$

On the r.h.s. of the equation for $\xi(z)$ we used the vector representation in the diagonal $U(1)$-charge Y basis, where Z is nilpotent $Z^2 = 0$. The normalization factor $-\frac{i}{2}$ in the definition of Z is chosen such that we get the simple matrix expression for $\xi(z)$ expressed in terms of z which coincides with (27). Notice the distinction between z and Z: Z is the linear combination of negatively charged broken generators $\bar{X}_{ij}$ contracted with the complex coordinates z^{ij} of the coset space. Therefore Z is represented by a $2N \times 2N$ matrix, while z is an $N \times N$ matrix. Using the projection operator η_- defined in eq. (32) and $\xi(z)$ the Kähler potential is given by (47) and (43)

$$K(z, \bar{z}) = \ln \det{}_{\eta_-} [\xi(-z)\xi^\dagger(-\bar{z})] = \ln \det \chi^{-1}, \qquad \chi^{-1} = \mathbb{1} + z\bar{z}. \tag{84}$$

Here the $\det{}_{\eta_-}$ denotes that the determinant is defined on the subspace on which the projection η_- acts as the identity. Notice that the submetric $\tilde{\chi}$ defined in (33) is the transposed $\tilde{\chi} = \chi^T$ of χ because of the anti-symmetry of z.

We next determine the non-linear transformations of the anti-symmetric coordinates z^{ij} under the finite $g \in SO(2N)$ transformation. From eq. (29) we know directly that

$$^g z = (\gamma + \delta z)(\alpha + \beta z)^{-1}. \tag{85}$$

The submetric χ transforms under these finite $SO(2N)$-transformations as

$$\chi(^g z, ^g \bar{z}) = (\hat{h}_-^\dagger)^{-1} \chi(z, \bar{z}) (\hat{h}_-)^{-1}, \qquad \hat{h}_-(z; g) = (\delta^{\dashv} - z\beta^{\dashv})^{-1}, \tag{86}$$

using eq. (30). Notice that according to eq. (82) $\hat{h}_+(z; g) = \hat{h}_-^T(z; g)$ is the transposed of $\hat{h}_-$, therefore we only use $\hat{h}_-$ in the following. The Kähler potential (84) transforms as follows

$$K(^g z, ^g \bar{z}) = K(z, \bar{z}) + F(z; g) + \bar{F}(\bar{z}; g), \tag{87}$$

where the holomorphic function $F(z; g)$ is given by

$$F(z; g) = \ln \det \hat{h}_-(z; g) \tag{88}$$

The complex Hermitean metric of the coset is obtained from the Kähler potential (84) in the standard way as the second mixed derivative

$$G_\sigma(dz, d\bar{z}) = \operatorname{tr}\left(dz\, \chi^T\, d\bar{z}\, \chi\right) = \operatorname{tr}\Big(dz\, \left(\mathbb{1} + \bar{z}z\right)^{-1} d\bar{z}\, (\mathbb{1} + z\bar{z})^{-1}\Big). \tag{89}$$

We next discuss the Killing potentials M_σ for the Goldstone scalar fields z and $\bar{z}$. The Killing potential M_σ, defined by eq. (14), can be written for the coset $SO(2N)/U(N)$ as

$$M_\sigma(u,x,\bar{x}) = \mathrm{Tr}(\Theta\tilde{M}_\sigma) = \mathrm{tr}(uM_{\sigma\,u} + xM_{\sigma\,x^\dagger} + x^\dagger M_{\sigma\,x}), \tag{90}$$

where the trace Tr is over $2N\times 2N$ matrices, while the trace tr is over $N\times N$ matrices. We have used a notation similar to eq. (80)

$$\Theta = \begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix} \quad \text{and} \quad \tilde{M}_\sigma = \begin{pmatrix} \tilde{M}_{\sigma\,u} & \tilde{M}_{\sigma\,x^\dagger} \\ -\tilde{M}_{\sigma\,x} & -\tilde{M}_{\sigma\,u^T} \end{pmatrix}, \tag{91}$$

so that $M_{\sigma\,x} = \tilde{M}_{\sigma\,x}$, $M_{\sigma\,x^\dagger} = \tilde{M}_{\sigma\,x^\dagger}$ and $M_{\sigma\,u} = \tilde{M}_{\sigma\,u} + \big(\tilde{M}_{\sigma\,u^T}\big)^T$. We now determine the Killing potentials explicitly. We will introduce some notation that might seem somewhat cumbersome at this stage, but which will be convenient when we discuss the Killing potentials due to additional matter coupling. Define the matrices R and R_T by

$$R(z;\Theta) = x - u^T z - zu + zx^\dagger z, \qquad R_T(z;\Theta) = -u^T + zx^\dagger. \tag{92}$$

Notice that $\delta z = R(z;\Theta)$ is a compact notation for the Killing vectors of the coset space, and $\mathrm{tr}R_T = F(z)$, the holomorphic Kähler transformation. Computing the Killing potentials M_σ in the standard way (14) gives

$$-iM_\sigma(z,\bar{z};\Theta) = -\mathrm{tr}\,\Delta(z,\bar{z};\Theta), \tag{93}$$

where we have defined the matrix Δ in analogy to the Killing potentials associated with the Grassmannian cosets [12] by

$$\Delta(z,\bar{z};\Theta) \equiv R_T - R\bar{z}\chi = (zu\bar{z} - u^T - x\bar{z} + zx^\dagger)\chi. \tag{94}$$

The matrix Δ can also be written in terms of the BKMU-variable $\xi(z)$ and Θ and the projector $\tilde{\eta}_-^T = \begin{pmatrix} 0 & \mathbb{1} \end{pmatrix}$ as

$$\Delta(z,\bar{z};\Theta) = \tilde{\eta}_-^T\,(\xi(z))^{-1}\,\Theta\,\big(\xi^\dagger(\bar{z})\big)^{-1}\,\tilde{\eta}_-\,\chi. \tag{95}$$

Using that $(\xi(z))^{-1} = \xi(-z)$, the Killing potential matrix $\tilde{M}_\sigma$ is given by

$$-i\tilde{M}_\sigma = -\xi^\dagger(-\bar{z})\tilde{\eta}_-\,\chi\,\tilde{\eta}_-^T\xi(-z) = -\begin{pmatrix} \bar{z}\chi z & -\bar{z}\chi \\ -\chi z & \chi \end{pmatrix}. \tag{96}$$

From this we can read off the Killing potentials $M_{\sigma\,x}, M_{\sigma\,x^\dagger}$ and $M_{\sigma\,u}$ to find

$$-iM_{\sigma\,x} = -\chi z, \quad -iM_{\sigma\,x^\dagger} = \bar{z}\chi, \quad -iM_{\sigma\,u} = -2\bar{z}\chi z + \mathbb{1}. \tag{97}$$

4.4 Matter coupling

In this section we discuss different types of matter couplings to the supersymmetric $SO(2N)/U(N)$ σ-model. As we only need the decomposition of the chiral spinor representation of $SO(2N)$ in completely anti-symmetric $SU(N)$ tensors in our construction of anomaly-free models later, we focus here primarily on these representations. We first introduce a matter representation x which transforms in the same way as a differential. Under a finite transformation (85) the real superfield x transforms as

$$ {}^{g}x = \hat{h}_{-}(z;g) x \hat{h}_{-}^{T}(z;g), \tag{98} $$

using that $\hat{h}_{+} = \hat{h}_{-}^{T}$. An invariant Kähler potential for x is given by

$$ K(x,\bar{x};z,\bar{z}) = \operatorname{tr}\left(x \chi^{T} \bar{x} \chi \right). \tag{99} $$

Below we discuss non-linear $SO(2N)$ realizations on the irreducible completely anti-symmetric $SU(N)$-tensor representations with p indices and arbitrary rescaling charge q. We denote these tensors by $T_{(p;q)}^{i_1 \ldots i_p}$, or without indices by $T_{(p;q)}$, when no confusion is possible. We interpret them as matter multiplets and construct their invariant Kähler potentials. To define their transformation properties we first consider a vector $T^{i} = T^{i}_{(1;0)}$ without a rescaling charge. It transforms as

$$ {}^{g}T = \hat{h}_{-}(z;g) T, \tag{100} $$

under finite non-linear $SO(2N)$ transformations (85). An invariant Kähler potential for the vector $T = T_{(1;0)}$ is given by

$$ K_{(1;0)} = \bar{T} \chi T = \bar{T}_{i}\, \chi^{i}{}_{j} T^{j}, \tag{101} $$

with the metric χ defined in eq. (84).

It is also possible to couple a singlet chiral multiplet S to the coset, which can be interpreted as a section of the minimal line bundle. It transforms as (65)

$$ {}^{g}S = e^{\frac{1}{2}F(z)} S = (\det \hat{h}_{-})^{\frac{1}{2}} S, \tag{102} $$

so that its Kähler potential

$$ K_{(0;1)} = S \bar{S} e^{-\frac{1}{2} K_{\sigma}} \tag{103} $$

is invariant. With this singlet S, we can rescale any given chiral multiplet, for example $T^{i}_{(1;q)} \equiv S^{q} T^{i}_{(1;0)}$ transforms as

$$ {}^{g}T_{(1;q)} = {}^{g}(S^{q} T_{(1;0)}) = e^{\frac{q}{2}F(z)} \hat{h}_{-} T_{(1;q)} = (\det \hat{h}_{-})^{\frac{q}{2}} \hat{h}_{-} T_{(1;q)}. \tag{104} $$

Since S is a section of the minimal line bundle over the coset $SO(2N)/U(N)$ the rescaling charge q is integer. The generalization of Kähler potential (101) is given by

$$K_{(1;q)} = \bar{T}_{(1;q)}\chi_{(1;q)}T_{(1;q)}, \tag{105}$$

with the modified metric

$$\chi_{(1;q)} = e^{-\frac{q}{2}K_\sigma}\chi = (\det\chi)^{\frac{q}{2}}\chi. \tag{106}$$

Now we construct completely anti-symmetric tensor representations of higher rank. By taking the completely anti-symmetric tensor products of a set of $SU(N)$ vectors $\{T_1^{i_1}, \ldots, T_p^{i_p}\}$ we obtain an $SU(N)$ tensor of rank p with rescaling charge q

$$T^A_{(p;q)} = T^{i_1\ldots i_p}_{(p;q)} \equiv \frac{1}{p!}S^q T_1^{[i_1} * \ldots * T_p^{i_p]}. \tag{107}$$

Here we have introduced the multi-index notation $A = (i_1 \ldots i_p)$ and $[\ldots]$ denotes the complete anti-symmetrization of the indices inside the brackets. In analogy to the transformations of $T_{(1;0)}$ and S we obtain

$${}^gT^{i_1\ldots i_p}_{(p;q)} = (\det\hat{h}_-)^{\frac{q}{2}}(\hat{h}_-)^{i_1}{}_{j_1}\ldots(\hat{h}_-)^{i_p}{}_{j_p}T^{j_1\ldots j_p}_{(p;q)}. \tag{108}$$

The Kähler potential for this tensor $T_{(p;q)}$ is the direct generalization [11] of the Kähler potentials for the vector (101) and singlet (103)

$$K_{(p;q)} = \bar{T}_{(p;q)B}\,G^B_{(p;q)\,A}\,T^A_{(p;q)} = \frac{1}{p!}\bar{T}_{(p;q)j_1\ldots j_p}\,e^{-\frac{q}{2}K_\sigma}\,\chi^{j_1}{}_{i_1}\ldots\chi^{j_p}{}_{i_p}\,T^{i_1\ldots i_p}_{(p;q)}, \tag{109}$$

with the generalized metric

$$G^B_{(p;q)\,A} = \frac{1}{p!}(\det\chi)^{\frac{q}{2}}\chi^{j_1}{}_{i_1}\ldots\chi^{j_p}{}_{i_p}. \tag{110}$$

The $SU(N)$ Levi-Civita tensor $\epsilon_{i_1\ldots i_N}$ is invariant under $SU(N)$ transformations. We can use it to defined an $SU(N)$ dual tensor $T_{(\overline{N-p};q)\,i_{p+1}\ldots i_N}$ with $N-p$ indices and rescaling charge q by

$$T_{(\overline{N-p};q)\,i_{p+1}\ldots i_N} \equiv \frac{1}{p!}T^{i_p\ldots i_1}_{(p;q)}\epsilon_{i_1\ldots i_N}, \tag{111}$$

which transforms under the finite transformation (85) as

$${}^gT_{(\overline{p};q)i_1\ldots i_p} = T_{(\overline{p};q)j_1\ldots j_p}(\hat{h}_-^{-1})^{j_1}{}_{i_1}\ldots(\hat{h}_-^{-1})^{j_p}{}_{i_p}(\det\hat{h}_-)^{1+\frac{q}{2}}. \tag{112}$$

Note, that as $\sqrt{G}$ is not holomorphic, we have prefered to absorb it in a redefinition of the metric, rather than in the definition of the dual. The power $1+\frac{q}{2}$ of

the $\det \hat{h}_-$ instead of $\frac{q}{2}$ arises because we have changed from $\hat{h}_-$ to its inverse at the expense of an additional factor of the determinant of $\hat{h}_-$. In our conventions tensors have superscript indices while dual tensors have subscript indices. Cleary, working with anti-symmetric tensors or dual tensors is equivalent. The invariant Kähler potential for a dual tensor is given by

$$K_{(\overline{p};q)} = T_{(\overline{p};q)\,A}\, G^A_{(\overline{p};q)\,B}\, \bar{T}^B_{(\overline{p};q)} \tag{113}$$

where the metric is given by

$$G^A_{(\overline{p};q)\,B} = \frac{1}{p!}(\det \chi)^{1+\frac{q}{2}} (\chi^{-1})^{i_1}{}_{j_1} \dots (\chi^{-1})^{i_p}{}_{j_p}. \tag{114}$$

In addition we can construct a matter reprensentation A that transforms in the adjoint of $SU(N)$. The index structure of this matrix is $A^i{}_j$ and in addition it is traceless $\mathrm{tr}A = A^i{}_i = 0$. It transformation properties under the full non-linear $SO(2N)$ symmetries takes the form

$$^gA = \hat{h}_-(z;g) A \hat{h}_-^{-1}(z;g). \tag{115}$$

This transformation rule can be obtained by defining A as the tensor product of a vector $T_{(1;0)}$ and a dual vector $T_{(\bar{1};-2)}$ with rescaling charge -2

$$A = T_{(1;0)} \otimes T_{(\bar{1};-2)}. \tag{116}$$

It is easy to see that this gives the right generalization of the $SU(5)$ adjoint by restricting $SO(2N)$ to an $U(5)$ transformation:

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix} \Longrightarrow {}^gA = (\alpha^T)^{-1} A \alpha^T. \tag{117}$$

Clearly, A does not transform under the $U(1)$ factor of $U(5)$. Notice that the condition that A be traceless, is respected by the transformation rule (115). The simplest invariant Kähler potential for this matter field A is

$$K_A = \mathrm{tr}\left(\chi A \chi^{-1} \bar{A}\right). \tag{118}$$

We next turn to a discussion of the contributions $M_{(p;q)}$ and $M_{(\bar{p};q)}$ to the Killing potentials for a tensor $T_{(p;q)}$ and a dual-tensor $\bar{T}_{(\bar{p};q)}$ of rank p with a rescaling charge q, respectively. As the Kähler potentials $K_{(p;q)}$ and $K_{(\bar{p};q)}$ are invariant, their contributions to the Killing potentials are obtained from

$$-iM_{(p;q)} = K_{(p;q),\,\alpha}\mathcal{R}^\alpha, \qquad -iM_{(\bar{p};q)} = K_{(\bar{p};q),\,\alpha}\mathcal{R}^\alpha, \tag{119}$$

where $\delta_i Z^\alpha = \mathcal{R}_i^\alpha$ denote the Killing vectors (cf. eq. (92))

$$\delta z = R,$$

$$\delta T^{i_1\dots i_p}_{(p;q)} = \sum_{r=1}^{p} (R_T)^{i_r}{}_j T^{i_1\dots j\dots i_p}_{(p;q)} + \frac{q}{2}\,\mathrm{tr}(R_T)\,T^{i_1\dots i_p}_{(p;q)}, \tag{120}$$

$$\delta T_{(\bar{p};q)i_1\dots i_p} = \sum_{r=1}^{p} T_{(\bar{p};q)i_1\dots j\dots i_p}(-R_T)^j{}_{j_r} + \Big(1+\frac{q}{2}\Big)\mathrm{tr}(R_T)\,T_{(\bar{p};q)\,i_1\dots i_p}.$$

They follow from expanding the finite transformations (85), (108) and (112) to first order in the infinitesimal parameters $u, x, x^\dagger$.

The Killing potential for a rank p tensor with rescaling charge q is given by

$$-iM_{(p;q)} = \bar{T}_{(p;q)B} G^B_{(p;q)C} \Delta^C_{(p;q)A} T^A_{(p;q)}, \tag{121}$$

where, using the notation (94),

$$\Delta^C_{(p;q)A} = \sum_{r=1}^{p} \delta^{k_1}{}_{i_1}\dots\Delta^{k_r}{}_{i_r}\dots\delta^{k_p}{}_{i_p} + \frac{q}{2}\mathrm{tr}\Delta\,\delta^{k_1}{}_{i_1}\dots\delta^{k_p}{}_{i_p}. \tag{122}$$

To obtain this result we have made the following steps. We first obtained the Killing potential for a rank 1 tensor (a vector) with rescaling charge zero. This result can easily be generalized to a rank p tensor with rescaling charge zero. Next we construct the Killing potential for a rank 0 tensor (a singlet) with an arbitrary rescaling charge. Finally we put all results together to obtain eq. (121). We can proceed similarly to obtain the Killing potential $M_{(\bar{p};q)}$ for a rank p dual tensor with a rescaling charge q. As the dualization has introduced a determinant $\det \hat{h}_-$ in the finite transformation (112), it is more convenient to first consider a rank p dual tensor with rescaling charge -2, which precisely cancels the determinant. To obtain the final result for a rank p dual-tensor with a rescaling charge q, we have to rescale the rank again, which introduces a factor $1+\frac{q}{2}$. Finally, the Killing potential reads

$$-iM_{(\bar{p};q)} = T_{(\bar{p};q)B} \Delta^B_{(\bar{p};q)C} G^C_{(\bar{p};q)A} \bar{T}^A_{(\bar{p};q)}, \tag{123}$$

with $\Delta^C_{(\bar{p};q)A}$ defined as

$$\Delta^B_{(\bar{p};q)C} = \sum_{r=1}^{p} \delta^{j_1}{}_{k_1}\dots(-\Delta)^{j_r}{}_{k_r}\dots\delta^{j_p}{}_{k_p} + \Big(1+\frac{q}{2}\Big)\mathrm{tr}\,\Delta\,\delta^{j_1}{}_{k_1}\dots\delta^{j_p}{}_{k_p}. \tag{124}$$

The infinitesimal form of the transformation of the adjoint matter field A is given by

$$\delta A = R_T A - A R_T = [R_T, A] \tag{125}$$

and the resulting Killing potential can be written as

$$-iM_A = \mathrm{tr}\left(\chi\Delta A\chi^{-1}\bar{A} - \chi A\Delta\chi^{-1}\bar{A}\right) = \mathrm{tr}\left(\chi[\Delta, A]\chi^{-1}\bar{A}\right). \tag{126}$$

4.5 Consistent $SO(2N)/U(N)$ spinor models

In this subsection we construct an anomaly-free model based on the spinor representation of $SO(2N)$ that contains the coordinates of the coset $SO(2N)/U(N)$. Only for a limited number of choices for N such a model satisfies the line bundle constraint.

A supersymmetric model built on the $SO(2N)/U(N)$ coset space is not free of anomalies by itself, as all the $\frac{1}{2}N(N-1)$ anti-symmetric coordinates z^{ij} and therefore also their chiral fermionic partners carry the same charge 4 in the standard normalization. To construct a consistent supersymmetric model around this coset one can try to embed the coordinates in an anomaly-free representation. All representations of $SO(2N)$ are anomaly-free, unless $SO(2N)$ is isomorphic to a non-anomaly-free unitary group. This happens for $SO(2) \cong U(1)$ and $SO(6) \cong SU(4)$, hence we disregard the cases $N = 1, 3$ below. In appendix B we derive this result by calculating the possible $U(1)$ anomalies of the chiral spinor representation. An $SO(2N)$ representation that branches to an anti-symmetric 2-tensor of $SU(N)$ is the chiral spinor representation of $SO(2N)$. The other $U(N)$ representations that arise from the spinor representation transform under the full $SO(2N)$ symmetries via non-linear transformations. For global consistency this means that these matter representations are sections of bundles. If one of these sections is a line bundle we run into the cocycle condition, which greatly restricts the freedom of charge assignments. In section 3 we have determined the section of the minimal line bundle over $SO(2N)/U(N)$. As the dimension $2N$ is even, the irreducible representations carry definite chirality; we show that it is sufficient to consider only the positive chiral spinor representation for our purpose of extending the coset to the spinor representation. After that we turn to the main result of this subsection: the cocycle condition only allows for a very restricted class of consistent $SO(2N)/U(N)$ spinor models: $N = 2, 5, 6, 8$.

As argued in appendix A, to construct a consistent model on $SO(2N)/U(N)$ using irreducible spinor representations, we need to identify the anti-symmetric coordinates z^{ij} of the coset space with an anti-symmetric 2-tensor of the branching of the spinor. We have the following two states ${\psi_2}^{ij}$ or $\psi_{\bar{2}\,ij}$ as possible candidates. According to eq. (170), appendix A, the charge of ${\psi_2}^{ij}$ is $N-4$; it has positive chirality. The charge of $\psi_{\bar{2}\,ij}$ is opposite and its chirality is $(-)^N$. Notice that for $N = 4$ we can never construct a consistent model using the spinor representations as the charges of ${\psi_2}^{ij}$ and $\psi_{\bar{2}\,ij}$ are zero, while the charge of the coordinate z^{ij} is non-zero. For N is even both ${\psi_2}^{ij}$ and $\psi_{\bar{2}\,ij}$ have the same chirality, hence they are in the same irreducible representation. The duality operation (171), appendix A, maps the positive chirality states into themselves. Therefore, for even N it is sufficient to consider only the state ${\psi_2}^{ij}$ as the candidate for the coordinates z^{ij} of the coset. For odd N the only odd length state that can be associated with the coordinates z^{ij} has length $N-2$, but it is dual to the state with length 2. Therefore, for all N it is sufficient to consider only the positive chirality spinor

representation and only the state ${\psi_2}^{ij}$ as candidate for the coordinates z^{ij} of $SO(2N)/U(N)$.

We next discuss the restriction that the consistency of the line bundle poses on the construction of anomaly-free extensions of cosets $SO(2N)/U(N)$ using the positive chirality spinors of $SO(2N)$. We remarked before that the case $N = 4$ does not work as the state ${\psi_2}^{ij}$ does not carry Y charge. Therefore we consider the cases $N = 2$ and $N \geq 5$ from now on. It was shown in section 3 that the minimal charge of the line bundle over the coset space $SO(2N)/U(N)$ is equal to N when the charge of the coordinates is taken to be 4. This is the normalization employed in our detailed discussion of the $SO(2N)$ algebra in sect. 4.1. As all states of a positive chirality spinor have an even number of indices, the tensor structure of these states can be obtained from completely anti-symmetric tensor products of the tangent vectors of the coset $SO(2N)/U(N)$ tensored with an integral power of the minimal line bundle. In particular the state $\psi_{(2p;q)}$ with length $2p$ and rescaled with the $q(p;N)$th power of the minimal line bundle has a charge $4p+Nq(p;N)$. For each p this charge should be proportional to the charge $N - 4p$ of the anti-symmetric tensor with $2p$ indices within the positive chirality spinor representation. Therefore we obtain the relation $\lambda(N-4p) = 4p+Nq(p;N)$ where $\lambda \in \mathbb{R}$ is a constant to be determined. Since the anti-symmetric tensor with 2 indices ($p = 1$) is identified with the coordinates z^{ij} of the coset, it does not have a rescaling charge, hence we find that $\lambda = \frac{4}{N-4}$. Solving for $q(p;N)$ gives

$$q(p;N) \;=\; q(0;N)\;(1-p) \;=\; \frac{4}{N-4}\,(1-p). \tag{127}$$

For consistency of the line bundle we need that $q(p;N)$ is an integer for all $0 \leq p \leq [N/2]$. Notice that $q(p;N)$ is integer whenever $q(0;N)$ is integer. $q(0;N)$ is only an integer if $N-4$ is a divider of 4, which implies that $N = 0, 2, 3, 5, 6, 8$. Of course $N = 0$ is impossible, and though the case $N = 3$ satisfies the line bundle quantization condition, it does not lead to an anomaly free model. Therefore the possible choices are:

N	2	5	6	8
$q(0;N) = \frac{4}{N-4}$	-2	4	2	1

The case of $N = 2$ is trivial in the sense that the coset is isomorphic to the simplest coset $SU(2)/U(1)$ (i.e., the 2-sphere) because $SO(4) \cong SU(2) \times SU(2)$. Notice that except for the last case $N = 8$ we only use squares of the minimal line bundle.

We finish this section by giving the Kähler potentials for the anomaly-free $SO(2N)/U(N)$ models based on the positive chiral spinor representation. The matter content is fixed by the discussion above: we need for each $0 \leq p \leq [N/2]$

a rank $2p$ completely anti-symmetric $SU(N)$ tensor with rescaling charge $q(p;N)$ given in eq. (127), except for $p=1$; this case corresponds to an anti-symmetric tensor with two indices, for which we take the coordinates of the $SO(2N)/U(N)$ coset itself. Using the Kähler potentials for the coset (84) and for anti-symmetric tensor representations with an arbitrary rescaling charge (109), we can express the Kähler potential for the complete system by

$$\mathcal{K} = \frac{1}{2} K_\sigma + \sum_{p=0,\ p\neq 1}^{[N/2]} K_{(\,2p;\ q(p;N)\,)}. \tag{128}$$

Here we have included a factor $\frac{1}{2}$ so as to get the standard normalization of the kinetic terms of the Goldstone boson fields. In section 5 we discuss the consistent $SO(10)/U(5)$-spinor model in detail. There we give the explicit expression for the Kähler potential, using dual tensors to reduce the number of indices.

5 Analysis of the SO(10)/U(5)-spinor model

In this section we apply the constructions presented so far to obtain anomaly-free $SO(10)/U(5)$-spinor models in the context of global supersymmetry. We study in particular the role of the potentials in determining the realization of internal symmetries and supersymmetry in the original σ-model, as well as in various gauged versions. We find some rather surprising results concerning the gauged versions of the model, implying that —in spite of selecting anomaly-free combinations of representations— it is not possible to gauge just any arbitrary global symmetry. In particular, we find that gauging the full global $SO(10)$ is not possible, whilst the consistency of gauging all or part of the linear $SU(5)\times U(1)$ symmetry depends crucially on the vacuum expectation values and choice of parameters in the model. We present strong arguments that the natural value of the Fayet-Iliopoulos parameter ξ in the models with a linear gauge group containing $U(1)$ is determined by the scale f of the σ-model, by a relation of the type $|\xi| f^2 \sim \mathcal{O}(1)$. It is not difficult to see that some of these results are valid beyond the particular model chosen. A more general and extensive discussion will be given elsewhere [29].

The choice for the model on $SO(10)/SU(5)\times U(1)$ is motivated by its fermionic field content, corresponding to one complete family of quarks and leptons, including a right-handed neutrino. This can be seen by looking at the $SU(5)$ representations of the chiral multiplets that the model contains: the coordinate multiplets Φ^{ij} form the $\underline{10}$ of $SU(5)$. The completely anti-symmetric tensor with 4 indices is equivalent to the $\underline{\bar{5}}$ with a relative $U(1)$ charge -3; we denote it by Ψ_i. And finally we have a singlet Ψ of $SU(5)$, with $U(1)$ charge $+5$.

We denote the full set of chiral superfields by $\Sigma^\alpha = (\Phi^{ij}, \Psi_i, \Psi)$, their physical components collectively by $(Z^\alpha, \psi^\alpha_L)$. The scalar components of the various $SU(5)$

representations are denoted by $Z^\alpha = (z^{ij}, k_i, h)$. In the absence of any local gauge couplings, the kinetic part of the lagrangian for the model is given in terms of real composite superfields $\mathcal{K}(\bar{\Sigma}, \Sigma)$ by the supersymmetric expression (8), which is equivalent to

$$\begin{aligned} \mathcal{L}_{\mathcal{K}} = \mathcal{K}(\bar{\Sigma}, \Sigma)|_D = -G_{\underline{\alpha}\alpha}(\partial^\mu \bar{Z}^{\underline{\alpha}} \partial_\mu Z^\alpha + \bar{\psi}_L^{\underline{\alpha}} \overleftrightarrow{\not{D}} \psi_L^\alpha - \hat{H}^{\underline{\alpha}} \hat{H}^\alpha) \\ + \tfrac{1}{2} R_{\underline{\alpha}\alpha\underline{\beta}\beta}(\bar{\psi}_R^\alpha \psi_L^\beta)(\bar{\psi}_L^{\underline{\alpha}} \psi_R^{\underline{\beta}}). \end{aligned} \tag{129}$$

In the present case, the Kähler potential from which the metric $G_{\underline{\alpha}\alpha}$ is derived, is given by (128)

$$\begin{aligned} \mathcal{K}(\bar{Z}, Z) &= \frac{1}{2} K_\sigma + K_{(0;4)} + K_{(\bar{1};-4)} \\ &= \frac{1}{2f^2} \ln \det \chi^{-1} + (\det \chi)^2 |h|^2 + (\det \chi)^{-1} \, k \chi^{-1} \bar{k}. \end{aligned} \tag{130}$$

with the submetric $\chi^{-1} = \mathbb{1} + f^2 z\bar{z}$ and $e^{f^2 K_\sigma} = (\det \chi)^{-1}$. This is the explicit form of eq. (128) in the $SO(10)/U(5)$ case, after rescaling the Goldstone fields z by the mass parameter $m_\sigma = 1/f$ which sets the scale of the σ-model. The auxiliary fields $\hat{H}^\alpha$ are defined as $\hat{H}^\alpha = H^\alpha - \Gamma^\alpha_{\beta\gamma} \bar{\psi}_R^\beta \psi_L^\gamma$ with the connection $\Gamma^\alpha_{\beta\gamma} = G^{\alpha\underline{\alpha}} G_{\underline{\alpha}\beta,\gamma}$.

It is of particular importance to have an explicit expression for the kinetic terms of the Goldstone fields, which are modified by the presence of the matter terms in the Kähler potential (130). Following the procedures of [10] and [12], the kinetic terms for the scalars z^{ij} and the quasi-Goldstone fermions ψ_L^{ij} are determined by the matter-extended Kähler metric

$$G_\sigma(x, \bar{x}) = G_{\sigma(ij)}{}^{(kl)} x^{ij} \bar{x}_{kl} = f^2 E \mathrm{tr}(x \chi^T \bar{x} \chi) + e^{f^2 K_\sigma} f^2 k x \chi^T \bar{x} \bar{k}, \tag{131}$$

where x^{ij} stands for components of the Goldstone superfield Φ^{ij} or their gradients, whilst $E = \frac{1}{2f^2} + e^{f^2 K_\sigma} k \chi^{-1} \bar{k} - 2e^{-2f^2 K_\sigma} |h|^2$. For some applications it is convenient to write this as

$$G_\sigma(x, \bar{x}) = \mathrm{tr}(x \chi^T \bar{x} \hat{\chi}), \qquad \hat{\chi} = f^2 E \chi + e^{f^2 K_\sigma} f^2 \bar{k} k. \tag{132}$$

Clearly, the physical requirement that the model be ghost-free implies that this metric has to be sign-definite. As χ is positive definite, and the second term proportional to $\bar{k}k$ is non-negative, positive definiteness of the metric is guaranteed if $E > 0$. For $E < 0$, there always are negative kinetic-energy ghosts. However, for $E = 0$ a more detailed analysis is required.

In particular, we note that

$$\det \hat{\chi} = f^{10} E^4 \left(E + e^{f^2 K_\sigma} k \chi^{-1} \bar{k} \right) \det \chi. \tag{133}$$

As a result, for $E = 0$ the metric has a four-fold zero eigenvalue. This implies the existence of four complex orthogonal eigenvectors of $\hat{\chi}$ with zero eigenvalue; one can then construct six independent (complex) anti-symmetric tensor zero-modes, of the form $x = vw^T - wv^T$, with v, w independent zero eigenvectors. Of course, if $k = 0$ at the same time, the whole metric $\hat{\chi}$ vanishes; then also the kinetic terms of all Goldstone fields and their fermion partners vanish.

As concerns mass-terms, we observe that for the model (130) it is not possible to construct an $SO(10)$-invariant superpotential. First, the non-linear transformations of the coordinates z exclude their appearance in an invariant expression. Next, there is no non-vanishing holomorphic $SU(5)$ invariant for k_i. Finally, as h transforms under $U(1)$ and there is no field that compensates for its transformation, it also cannot appear in the superpotential. In the absence of a superpotential, all fields in the action (129) —the Goldstone bosons and their superpartners as well as the chiral superfields defining the matter representations— describe massless spin-0 and chiral spin-1/2 particles.

This situation changes if we add a second family of quarks and leptons, with superfields $\Sigma_{(2)} = (\Phi^{ij}_{(2)}, \Psi_{(2)\,i}, \Psi_{(2)})$. It is then possible to construct an invariant superpotential

$$W(\Sigma) = \sum_{a=1,2} \lambda_a \, \Psi_{(a)} \, \Psi_{(1)\,i} \Psi_{(2)\,j} \Phi^{ij}_{(2)}. \tag{134}$$

The λ_a are coupling constants of dimension $(\text{mass})^{-1}$.

As a next step towards a physical interpretation of the fermions as describing quarks and leptons, we introduce gauge interactions. This can have important implications for the spectrum of the theory, as in supersymmetric theories gauge-couplings are accompanied by Yukawa couplings and a D-term potential. We first consider gauging the full $SO(10)$. A local transformation of the form (85) then always allows one to go to the unitary gauge $z = \bar{z} = 0$. Thus all Goldstone bosons disappear from the spectrum as a result of the Brout-Englert-Higgs effect; this is confirmed by the finite mass-terms for the gauge fields corresponding to the broken generators of $SO(10)$.

However in the presence of matter fields as in (130), required for the cancellation of anomalies, the analysis of the D-terms in the potential shows that in the unitary gauge the model becomes singular: in the minimum of the potential the expectation value of the Kähler metric vanishes: $E = 0$ and $k = 0$. Thus the kinetic energy terms of the Goldstone and quasi-Goldstone fields all vanish. Actually, this seems to happen in other fully gauged supersymmetric σ-models on Kähler cosets with anomalies cancelled by matter as well.

As an alternative to gauging $SO(10)$, one can gauge only the linear subgroup $SU(5) \times U(1)$ instead. This explicitly breaks the non-linear global $SO(10)$. It is then allowed in principle to construct superpotentials which are invariant only under the local gauge symmetry, although one would expect the strength of this

potential to be proportional to the gauge coupling constant. In fact, this happens automatically with the D-term potentials. In addition, when gauging any group containing the $U(1)$ as a factor, the introduction of a Fayet-Iliopoulos term is allowed. It turns out, that the corresponding models are indeed well-behaved for a range of non-zero values of this parameter.

We now present details of this analysis. The theory defined by the Lagrangian (129), (130) has a global $SO(10)$ symmetry. This global symmetry allows vector bosons to be coupled to the model by turning the $SO(10)$ group, or its subgroup $SU(5) \times U(1)$, into a local gauge group by introducing covariant derivatives into the Lagrangian. The covariant derivatives are defined by:

$$\begin{aligned} \mathcal{D}_\mu Z^\alpha &= \partial_\mu Z^\alpha - A_\mu^i \mathcal{R}_i^\alpha, \\ \mathcal{D}_\mu \psi_L^\alpha &= \partial_\mu \psi_L^\alpha - A_\mu^i \mathcal{R}_{i,\beta}^\alpha \psi_L^\beta + \mathcal{D}_\mu Z^\gamma \Gamma_{\gamma\beta}^\alpha \psi_L^\beta. \end{aligned} \tag{135}$$

Here the A_μ^i are the gauge fields corresponding to the local symmetries. They are components of the vector multiplets $V^i = (A_\mu^i, \lambda^i, D^i)$, with λ^i representing the gauginos and D^i the real auxiliary fields. The isometries $\mathcal{R}_i^\alpha$ are generated by Killing vectors as in eq. (120); here they take the form

$$\delta_\Theta z = R, \qquad \delta_\Theta h = 2\mathrm{tr}(R_T)h, \qquad \delta_\Theta k = -k(R_T + \mathrm{tr}(R_T)\mathbb{1}), \tag{136}$$

with $R(z;\Theta) = \frac{1}{f}\, x - u^T z - zu + f\, zx^\dagger z$ and $R_T(z;\Theta) = -u^T + f\, zx^\dagger$. Adapting its normalization to that of the kinetic terms (130), the full Killing potential generating these Killing vectors is $\mathcal{M} = \frac{1}{2}M_\sigma + M_{(0;4)} + M_{(\bar{1};-4)}$, which takes the explicit form (cf. eq.(94)):

$$-i\mathcal{M} = \mathrm{tr}\Delta(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}) - e^{f^2 K_\sigma} k \Delta \chi^{-1} \bar{k}. \tag{137}$$

After introduction of the gauge fields in lagrangian (129), via the covariant derivatives (135), the σ-model itself is no longer invariant under supersymmetry transformations. Supersymmetry is restored by adding terms

$$\Delta\mathcal{L}_\mathcal{K} = 2G_{\alpha\underline{\alpha}}(\mathcal{R}_i^\alpha \bar{\psi}_L^{\underline{\alpha}} \lambda_R^i + \bar{\mathcal{R}}_i^{\underline{\alpha}} \bar{\lambda}_R^i \psi_L^\alpha) - D^i(\mathcal{M}_i + \xi_i). \tag{138}$$

We have added a Fayet-Iliopoulos term with parameter ξ_i in case there is a commuting $U(1)$ vector multiplet. The full lagrangian for this model after introducing gauge interactions becomes

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{chiral}, \tag{139}$$

where $\mathcal{L}_{YM}$ is the usual supersymmetric Yang-Mills action, of the generic form

$$\mathcal{L}_{YM} = -\frac{1}{g^2}\,\mathrm{Tr}(\frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \frac{1}{2}\bar{\lambda}\not{D}\lambda - \frac{1}{2}D^2). \tag{140}$$

We use Tr to denote a trace over $2N \times 2N$-matrices; in contrast, traces over $N \times N$-matrices are denoted by tr. When gauging a product of several commuting subgroups of G, e.g. $SU(5) \times U(1)$, there is a coupling constant g_i for each of the subgroup factors. $\mathcal{L}_{chiral}$ is given by (129), but with ordinary derivatives $\partial_\mu Z^\alpha$, $\partial_\mu \psi_L^\alpha$ replaced by the covariant derivatives (135), while adding $\Delta\mathcal{L}_\mathcal{K}$:

$$\mathcal{L}_{chiral} = \mathcal{L}_\mathcal{K}(\partial_\mu \to \mathcal{D}_\mu) + \Delta\mathcal{L}_\mathcal{K}. \tag{141}$$

Next we analyze the scalar potential obtained by elimination of the D-fields for various gaugings. By substituting the expression (94) for Δ we obtain in index-free notation[1]

$$\begin{aligned}
-i\mathcal{M}_u &= (\mathbb{1} - 2f^2\bar{z}\chi z)\Big(\frac{1}{2f^2} + K_{(\bar{1};-4)} - 2K_{(0;4)}\Big) + e^{f^2K_\sigma}(k^T\bar{k}^T - f^2\bar{z}\bar{k}kz),\\
-i\mathcal{M}_{x^\dagger} &= -f\bar{z}\chi\Big(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}\Big) + fe^{f^2K_\sigma}\bar{z}\bar{k}k,\\
-i\mathcal{M}_x &= f\chi z\Big(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}\Big) - fe^{f^2K_\sigma}\bar{k}kz.
\end{aligned} \tag{142}$$

If the full $SO(10)$ is gauged, the unitary gauge can be chosen in which all Goldstone bosons $(z, \bar{z})$ vanish. This implies that the broken Killing potentials $\mathcal{M}_x$ and $\mathcal{M}_{x^\dagger}$ vanish automatically, leaving us with the $U(5)$ Killing potentials only. If we only gauge $U(5)$ then the Killing potentials $\mathcal{M}_x$ and $\mathcal{M}_{x^\dagger}$ are irrelevant, and again we have to consider only the $U(5)$ Killing potentials. However, in this case z represents a physical degree of freedom, and its vacuum expectation value does not necessarily vanish: $\langle z\rangle = 0$ is guaranteed only if $SU(5)$ is not broken.

To analyze both gauged $SO(10)$ and gauged $SU(5)\times U(1)$ at once, we consider the D-term potential arising from the gauging of $SU(5)\times U(1)$ including a Fayet-Iliopoulos term with parameter ξ for the $U(1)$:

$$V = \frac{g_1^2}{2N}(\xi - i\mathcal{M}_Y)^2 + \frac{g_5^2}{2}\mathrm{tr}(-i\mathcal{M}_t)^2. \tag{143}$$

Here the $U(5)$ Killing potentials $\mathcal{M}_Y$ and $\mathcal{M}_t$ are trace and the traceless part of $\mathcal{M}_u$:

$$\mathcal{M}_t = \mathcal{M}_u - \frac{1}{N}\mathcal{M}_Y\,\mathbb{1}, \qquad \mathcal{M}_Y = \mathrm{tr}\mathcal{M}_u. \tag{144}$$

We can derive $\mathrm{tr}\mathcal{M}_t^2$ from $\mathcal{M}_Y$ and $\mathrm{tr}\mathcal{M}_u^2$ by

$$\mathrm{tr}(-i\mathcal{M}_t)^2 = \mathrm{tr}(-i\mathcal{M}_u)^2 - \frac{1}{N}(-i\mathcal{M}_Y)^2. \tag{145}$$

[1]The factors $-i$ here result from Δ being anti-hermitean, eq.(93).

An explicit expression for $\mathcal{M}_t$ in terms of the matter-extended submetric $\hat{\chi}$ is

$$-i\mathcal{M}_t = \frac{2}{f^2}\hat{\chi}^T - 2\gamma\mathbb{1} - e^{f^2K_\sigma}\left(f^2\bar{z}\bar{k}\,kz + k^T\bar{k}^T\right), \tag{146}$$

where γ is defined by

$$N\gamma = (\mathrm{tr}\chi)E + \frac{1}{2}e^{f^2K_\sigma}k(1-f^2z\bar{z})\bar{k}. \tag{147}$$

The terms in the potential (143) are proportional to the square of the coupling constants g_1 and g_5 of the $U(1)$ and $SU(5)$ gauge groups, respectively. The case of fully gauged $SO(10)$ is reobtained by taking the coupling constants equal: $g_1 = g_5 = g_{10}$, and the Fayet-Iliopoulos term to vanish: $\xi = 0$. We have left the rank $N = 5$ of $SO(10)$ in, so as to keep track of some of the dependence on this rank.

The potential (143) is non-negative. In order for supersymmetry to be preserved, the minimum must be at $V_{min} = 0$; in contrast, $V_{min} > 0$ implies spontaneous supersymmetry breaking by the potential. Being a sum of squares, a vanishing potential is possible only if $\mathcal{M}_Y = 0$ and $\mathcal{M}_t = 0$ at the same time.

In the case of gauged $SO(10)$ one can always work in the unitary gauge $z = \bar{z} = 0$. However, in the case of gauged $SU(5) \times U(1)$ the potential can cause further symmetry breaking by generating a vacuum expectation value for the would-be Goldstone bosons. Because of its antisymmetry, an $SU(5) \times U(1)$ transformation can be performed to put $\langle z\rangle$ into the standard form

$$\langle fz\rangle = \begin{pmatrix} a\sigma_2 & & \\ & b\sigma_2 & \\ & & 0 \end{pmatrix}, \tag{148}$$

with real $a, b \geq 0$. Of course, the unitary gauge is included as the special case $a = b = 0$. The vacuum expectation value (148) preserves a subgroup $SU(2) \times SU(2) \times U(1)$. If the $\underline{\bar{5}}$ gets a vacuum expectations value, this residual symmetry can be used to chose

$$\langle k\rangle = (k_1, 0, k_3, 0, k_5)\,. \tag{149}$$

We first investigate the existence of zeros of the potential, compatible with supersymmetry. The condition $\langle\mathcal{M}_t\rangle = 0$ then implies $k_1 = k_3 = 0$, and

$$E = \gamma\left(1+a^2\right) = \gamma\left(1+b^2\right) = \gamma - \left(1+a^2\right)\left(1+b^2\right)|k_5|^2. \tag{150}$$

There are three separate solutions to these conditions; the first is

$$a = b = k_5 = 0, \qquad E = \gamma. \tag{151}$$

This solution includes the unitary gauge. A second solution (which coincides with the previous one for $a = b = 0$) is the case $E = \gamma = k_5 = 0$. It can be seen immediately to yield $\hat{\chi} = 0$. Therefore in this case the kinetic terms of the Goldstone superfield components vanish. Such a solution is unacceptable, not only because part of the quarks and leptons disappear from the spectrum of physical states, but even more importantly as this upsets the cancellation of anomalies, which is guaranteed only if all chiral fermions in the model contribute. This holds in particular for the case of fully gauged $SO(10)$ in the unitary gauge.

The third solution of the supersymmetric vacuum conditions, which exists only for $E, \gamma < 0$, is

$$a^2 = b^2 = -\frac{1}{\gamma}\left(1+a^2\right)^4 |k_5|^2, \qquad E = \gamma\left(1+a^2\right). \tag{152}$$

Inserting this solution into the expression (132) one obtains $\hat{\chi} = f^2\gamma\mathbb{1}$, which in this case is negative definite. As it is not possible to change the overall sign of the Kähler potential without creating negative kinetic energy terms for the matter fields, this solution always contains ghosts and is again physically unacceptable.

The upshot of this discussion is, that physically consistent models (i.e. anomaly-free, with positive definite kinetic energy), in which the potential has zeros, require $z = k = 0$ and $E = \gamma > 0$. Such models can be realized with gauged $SU(5) \times U(1)$, but the model with fully gauged $SO(10)$ is excluded. We observe, that positivity of E for these solutions implies

$$0 \leq |h|^2 < \frac{1}{4f^2}. \tag{153}$$

Thus, unless $h = 0$, these solutions always spontaneously break $U(1)$, whilst $SU(5)$ is manifestly preserved.

The conditions (151) and (153) are necessary to have physically consistent models with $\langle \mathcal{M}_t \rangle = 0$. This is sufficient for a zero of the potential in a model with gauged $SU(5)$ only. If $U(1)$ is gauged, a zero of the potential requires the additional condition:

$$\begin{aligned} \xi = \langle i\mathcal{M}_Y \rangle = & -\left(1+a^2\right)^2\left(1+b^2\right)^2\left[\left(1-a^2\right)|k_1|^2 + \left(1-b^2\right)|k_3|^2 + |k_5|^2\right] \\ & -\left(1 + 2\frac{1-a^2}{1+a^2} + 2\frac{1-b^2}{1+b^2}\right) E. \end{aligned} \tag{154}$$

Combining this with $z = k = 0$, it follows that (with $N = 5$)

$$E = \gamma = -\frac{\xi}{N}, \qquad |h|^2 = \frac{1}{4f^2} + \frac{\xi}{2N}. \tag{155}$$

A consistent solution of this type exists only for $-N/(2f^2) \leq \xi < 0$. The kinetic energy for the Goldstone superfield components is now proportional to $(-f^2\xi)/N$.

Clearly, for values of ξ in this range it is necessary to perform a finite renormalization of the Goldstone superfields to obtain the canonical value of the kinetic terms; in the Kähler potential this is equivalent to a rescaling of the σ-model scale such that $f^2 \to -N/\xi$. In these models the natural value of the Fayet-Iliopoulos-parameter is therefore the σ-model scale, thereby relating internal and supersymmetry breaking.

We finish this section by observing that, in addition to zeros of the potential, there can also be ranges of the parameters (g_1^2, g_5^2, ξ), or models with only some proper subgroup of $SU(5)$ gauged, for which the minimum of the potential occurs at a positive value: $\langle V \rangle > 0$. In this case supersymmetry is manifestly broken by the potential. This could happen for example in the domain $\xi < -N/(2f^2)$. However, we have not performed an exhaustive analysis of this case.

6 Conclusion

In this paper we have considered supersymmetric models based on classic Kählerian coset spaces: $U(M+N)/U(M)\times U(N), USp(2N)/U(N)$ and $SO(2N)/U(N)$, and their non-compact versions. Starting from a non-linear realization of the group $SL(N+M, \mathbb{C})$ in finite form, we constructed their Kähler potentials. A generalization of the Killing potential for finite transformations has been obtained. The Kähler potential of such a coset can be written as a function of a fundamental submetric. This submetric also allows us to construct Kähler potentials for superfields as sections of bundles over the original classical coset. For most of these matter representations the naive definitions are sufficient to guarantee the existence of these bundles globally. However, the consistency of line bundles requires that the cocycle condition is satisfied.

We have discussed various aspects of these general constructions for classical Kählerian coset space in more detail for the class of orthogonal cosets $SO(2N)/U(N)$. All supersymmetric matter fields which form completely anti-symmetric representations of $SU(N)$ with arbitrary integer charges satisfying the cocycle condition have been obtained explicitly.

Pure supersymmetric coset models are often anomalous due to their chiral fermions. This is also the case for orthogonal cosets $SO(2N)/U(N)$, but as all $SO(2N)$ representations are anomaly free (with the exceptions of $SO(2) \cong U(1)$ and $SO(6) \cong SU(4)$), the supersymmetric field content can be extended such that all anomalies cancel. The completely anti-symmetric $SU(N)$ representations descending from the positive-chirality spinor representation of $SO(2N)$ provide possible candidates for anomaly free models, which can include the Goldstone bosons. However, the $U(1)$ charges of these anti-symmetric representations can often not be realized using the bundles at our disposal. In fact, only for $N = 2, 5, 6, 8$ these

$SO(2N)/U(N)$-spinor models can fulfil the consistency requirements of the line bundle.

Some phenomenological aspects of the $SO(10)/U(5)$-spinor model have been investigated. This model contains the $SU(5) \times U(1)$ fermionic field content of one generation of quarks and leptons, including a right-handed neutrino. The matter-extended metric for the Goldstone bosons of the coset is not automatically positive definite. In order that the theory is ghost-free when expanded around a minimum of the potential, the quantity E has to be positive, see eq. (132). The consequences of this physical requirement have been analysed for supersymmetric minima, if part of the isometry group is gauged. If the whole $SO(10)$ is gauged, the analysis is straightforward as one can employ the unitary gauge to put the Goldstone bosons to zero. We find the kinetic energy of the would-be Goldstone modes and their fermionic partners to vanish. Therefore the quasi-Goldstone fermions no longer contribute to the cancellation of anomalies.

Gauging (part of) the linear subgroup $U(5)$ calls for a more involved investigation. First we have obtained all supersymmetric minima for the case where $SU(5)$ is gauged. We found three classes of such vacua, of which two are physically problematic as the kinetic terms of the Goldstone multiplets either vanish or have negative values. The third type of supersymmetric vacuum only exists for a finite range of vacuum expectation values of the scalar partner of the right-handed neutrino. If the $U(1)$ factor is gauged in addition, the Fayet-Iliopoulos parameter is related directly to the vacuum expectation value of this scalar. This shows that only for a finite range of values of the Fayet-Iliopoulos parameter $U(5)$ can be gauged consistently.

A Decomposition of $SO(2N)$ spinors into anti-symmetric tensors

An arbitrary spinor $\boldsymbol{\psi}$ of $SO(2N)$ can be represented using anti-symmetric tensors $\psi_{p\, i_1 \dots i_p}$ of $SU(N)$ with p indices as

$$\boldsymbol{\psi} = \left(\psi_0, \psi_1^{i_1}, \dots, \psi_N^{i_1 \dots i_N} \right). \tag{156}$$

The invariant inner-product of two spinors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\psi}^\dagger \boldsymbol{\phi} = \sum_{p=0}^{N} \frac{1}{p!} \psi^\dagger_{p\, i_p \dots i_1} \phi_p{}^{i_1 \dots i_p} \tag{157}$$

where $\psi^\dagger_{p\, i_p \dots i_1} = \psi_p^{*\, i_1 \dots i_p}$. We want to construct a basis for the anti-symmetric $SU(N)$-tensors, and also a basis for the $SO(2N)$-spinors, using the Clifford algebra of fermion creation and annihilation operators Γ^i and $\bar{\Gamma}_i$, as introduced by

R.N. Mohapatra and B. Sakita [24], see also [27] and [25]. They satisfy the usual anti-commutation relations

$$\{\Gamma^i, \bar{\Gamma}_j\} = \delta^i{}_j, \qquad \{\Gamma^i, \Gamma^j\} = \{\bar{\Gamma}_i, \bar{\Gamma}_j\} = 0. \tag{158}$$

Assume that we have constructed a Hilbert space on which these Clifford operators act. In this Hilbert space we define the vacuum state $|0\rangle$ by $\Gamma^i|0\rangle = 0$ for any i. The ket- and bra-states

$$\mathbf{e}_{p\, i_1 \dots i_p} = \bar{\Gamma}_{i_1} \dots \bar{\Gamma}_{i_p}|0\rangle \qquad \mathbf{e}_p^{\dagger\, i_p \dots i_1} = \langle 0|\Gamma^{i_p} \dots \Gamma^{i_1} \tag{159}$$

satisfy the orthonomality relations

$$\mathbf{e}_p^{\dagger\, i_p \dots i_1}\mathbf{e}_{q\, j_1 \dots j_q} = 0, \quad \text{for} \quad p \neq q \quad \text{and} \quad \mathbf{e}_p^{\dagger\, i_p \dots i_1}\mathbf{e}_{p\, j_1 \dots j_p} = \delta^{i_1}_{[j_1} \dots \delta^{i_p}_{j_p]}, \tag{160}$$

where $\delta^{i_1}_{[j_1} \dots \delta^{i_p}_{j_p]}$ is the complete anti-symmetrized Kronecker-delta. Therefore the states $\mathbf{e}_{p\, i_1 \dots i_p}$ form a basis of anti-symmetric rank p tensors of $SU(N)$. Using the complete anti-symmetry it is easy to show that the number of the vectors $\mathbf{e}_p$ with length p is equal to $\binom{N}{p}$, hence the total number of vectors $\{\mathbf{e}_p\}$ is equal to 2^N. The collection of these states $\mathbf{e}_p$ for $0 \leq p \leq N$ form a basis for $SO(2N)$-spinors, hence $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^\dagger$ can be expanded in this basis

$$\boldsymbol{\psi} = \sum_{p=0}^{N} \frac{1}{p!}\psi_p{}^{i_1 \dots i_p}\mathbf{e}_{p\, i_1 \dots i_p} \quad \text{and} \quad \boldsymbol{\psi}^\dagger = \sum_{p=0}^{N} \frac{1}{p!}\psi^\dagger_{p\, i_p \dots i_1}\mathbf{e}_p^{\dagger\, i_1 \dots i_p}. \tag{161}$$

It is straightforward to check that in this basis the inner-product of two spinors $\boldsymbol{\psi}^\dagger\boldsymbol{\phi}$ is consistent with the definition (157) using the Clifford properties (158).

In terms of the Clifford algebra, we define the $2N$ gamma-matrices Γ_a with $a = 1, \dots, 2N$ by

$$\Gamma_a = \begin{cases} i(\Gamma^i - \bar{\Gamma}_i), & a = i = 1, \dots, N, \\ \Gamma^i + \bar{\Gamma}_i, & a = i + N = N+1, \dots, 2N, \end{cases} \tag{162}$$

with the property

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \tag{163}$$

This property can be used to show that the sigma-matrices

$$M_{ab} = \frac{1}{2}\Sigma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b], \tag{164}$$

are the generators of the $SO(2N)$-algebra (68) in the spinor representation. With respect to the spinor inner-product (157) the gamma-matrices are Hermitean

$\Gamma_a^\dagger = \Gamma_a$ and hence the sigma-matrices are anti-Hermitean $\Sigma_{ab}^\dagger = -\Sigma_{ab}$. Furthermore it implies that w.r.t. this inner product the fermion creation/annihilation operators are Hermitean conjugates:

$$\left(\Gamma^i\right)^\dagger = \bar{\Gamma}_i, \qquad \left(\bar{\Gamma}_i\right)^\dagger = \Gamma^i. \tag{165}$$

For products of Clifford operatros A and B we have $(AB)^\dagger = B^\dagger A^\dagger$.

The Hermitean chirality operator $\tilde{\Gamma}$ defined by

$$\tilde{\Gamma} = (-)^{\frac{1}{2}N(N-1)} i^{-N} \prod_{a=1}^{2N} \Gamma_a \tag{166}$$

can be written in terms of the Clifford elements as

$$\tilde{\Gamma} = \prod_i [\Gamma^i, \bar{\Gamma}_i] = \prod_{i=1}^{N} (1 - 2\hat{n}_i) = (-)^{\hat{n}}. \tag{167}$$

Here we have defined the ith number operator $\hat{n}_i = \bar{\Gamma}_i \Gamma^i$ and the total number operator $\hat{n} = \sum_i \hat{n}_i$. Using this chirality operator, we can define positive and negative chirality spinors in $2N$ dimensions

$$\tilde{\Gamma} \boldsymbol{\psi}_\pm = \pm \boldsymbol{\psi}_\pm \tag{168}$$

Using the form of the chirality operator (167), it follows that the positive chirality components of a spinor are given by completely anti-symmetric $SU(N)$-tensors of even length p, while the negative chirality components have odd length p.
The generators $U^i{}_j$ can be expressed in terms of the fermion operators as

$$U^i{}_j = -\frac{1}{2} [\Gamma^i, \bar{\Gamma}_j], \tag{169}$$

and satisfy the $U(N)$ algebra (69). Their anti-symmetric part $A^i{}_j$, and symmetric part $S^i{}_j$, take the form

$$A^i{}_j = -\tfrac{1}{4} \left([\chi^i, \bar{\chi_j}] - [\chi^j, \bar{\chi_i}] \right), \quad S^i{}_j = \tfrac{i}{4} \left([\chi^i, \bar{\chi_j}] + [\chi^j, \bar{\chi_i}] \right)$$

Furthermore the broken $SO(2N)$ generators X^{ij} and $\bar{X}_{ij}$ can be represented by

$$X^{ij} = \Gamma^i \Gamma^j \quad \text{and} \quad \bar{X}_{ij} = \bar{\Gamma}_i \bar{\Gamma}_j.$$

The $U(1)$-charge operator (73) is given in terms of the total number operator $\hat{n}$ by

$$Y = \sum_i [\Gamma^i, \bar{\Gamma}_i] = N - 2\hat{n}, \tag{170}$$

hence the charge of an anti-symmetric tensor with p indices, that occurs in the decomposition of a spinor, is $N-2p$: $Y\mathbf{e}_p = (N-2p)\mathbf{e}_p$. We define the dual vectors $\mathbf{e}_{\overline{N-p}}$ resp. $\mathbf{e}^\dagger_{\overline{N-p}}$ of the basis vectors $\mathbf{e}_p$ and $\mathbf{e}^\dagger_p$ resp. by

$$\mathbf{e}_{\overline{N-p}}{}^{i_N\ldots i_{p+1}} = \frac{1}{p!}\epsilon^{i_N\ldots i_1}\mathbf{e}_{p\,i_1\ldots i_p} \quad \text{and} \quad \mathbf{e}^\dagger_{\overline{N-p}\,i_{p+1}\ldots i_N} = \frac{1}{p!}\mathbf{e}^{\dagger\,i_p\ldots i_1}_p\epsilon_{i_1\ldots i_N}. \tag{171}$$

For the components ψ_p we use analogous definitions. Notice that under dualization the charge does not change, only the number of indices does.

B Anomaly cancellation of the spinor representation

We now show that the positive chirality spinors of $SO(2N)$ have no pure $U(1)$-anomaly, unless $SO(2N)$ is isomorphic to a non-anomaly-free unitary group, $SO(2) \cong U(1)$ or $SO(6) \cong SU(4)$, by computing the possible $U(1)$-anomalies. However it is straightforward to also compute the $U(1)$-anomalies for negative chiralities, so we calculate both here. The Y^k-anomaly $A_\pm(Y^k;N) = \mathrm{Tr}_\pm Y^k$ for the $\pm$ chirality spinor representation is given by

$$A_\pm(Y^k;N) = \sum_{l=0}^{N}\binom{N}{l}\frac{1\pm(-)^l}{2}(N-2l)^k. \tag{172}$$

This follows using the multiplicities $\binom{N}{l}$ and charges $N-2l$ of the states $\psi_l{}^{i_1\ldots i_l}$. The factor $\frac{1\pm(-)^l}{2}$ is introduced to project onto the positive or negative chirality states. The necessary details to obtain these results can be found in appendix A. To calculate these anomalies it is convenient to introduce the functions

$$q_\pm(x) = \pm\frac{1}{x} + x \tag{173}$$

of a variable x, in terms of which we define

$$P_\pm(x;N) \equiv \frac{1}{2}\left[(q_+)^N \pm (q_-)^N\right] = \sum_{l=0}^{N}\binom{N}{l}\frac{1\pm(-)^l}{2}x^{N-2l}. \tag{174}$$

Notice that the charge operator Y can be represented by $Y = x\frac{d}{dx}$. The anomaly $A_\pm(Y^k)$ can be calculated using the functions $P_\pm$ by

$$A_\pm(Y^k;N) = \left(x\frac{d}{dx}\right)^k P_\pm(x;N)\Bigg|_{x=1}. \tag{175}$$

To compute this we use the properties of the functions $q_\pm$

$$x\frac{d}{dx}q_\pm = q_\mp, \quad q_+|_{x=1} = 2, \quad \text{and} \quad q_-|_{x=1} = 0. \tag{176}$$

We obtain the following results for the Y and Y^3 anomalies in $D = 4$ dimensions

$$A_\pm(Y;N) = \begin{cases} \pm 1 & N = 1, \\ 0 & N \neq 1, \end{cases} \tag{177}$$

and

$$A_\pm(Y^3;N) = \begin{cases} \pm 3!2^2 & N = 3, \\ \pm 1 & N = 1, \\ 0 & N \neq 1, 3. \end{cases} \tag{178}$$

Hence we see that the cases $N = 1$ and $N = 3$ have indeed an anomalous spinor representation. We conclude from this anomaly analysis that for $N = 2$ and $N \geq 4$ the spinor representation of $SO(2N)$ is $U(1)$ anomaly-free.

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